March 9, 2025 Math 8

<u>**Handout 5: Recap. Practical probabilities. Poker hands.**</u>

In the game of poker, a player is dealt five cards (a "hand") from a regular deck with 4 suits (♠,♣,♦,♥) with card values in the following order: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A. What are the probabilities of the following combinations:

Royal Flush: 10, J, Q, K, A of any suit (Example: 10♥, J♥, Q♥, K♥, A♥) There are only 4 of them.

Straight Flush: Five cards in a row of the same suit (Example: 6♥, 7♥, 8♥, 9♥, 10♥). Each of these can start from any card from A to 9 and be in each of the for suits: $9 \times 4 = 36$. Notice that we excluded royal flushes from out computation (if we start with 10, we get a Royal Flush).

Four of a kind: Four cards of the same value, and one other random card (Example: K♥, K♠, K♦, K♣, 2♣). Which card 13×, which other value 12×, which another suit 4, total $= 13 \cdot 12 \cdot 4$.

Full House: Three cards of the same value, and two cards of the same value (Example: K♥, K♠, K♦, 4♠, 4♣). Which card for 3 13×, which three suits $\binom{4}{3}\times$, which card for a pair 12×, which two suits $\binom{4}{2}\times$, total $= 13\binom{4}{3}\,12\binom{4}{2}$.

Flush: Five cards of the same suit, not in order (Example: 3♥, 6♥, 8♥, J♥, A♥). Which suit 4×, which five cards $\binom{13}{5}\times$, total $= 4\binom{13}{5}$. However, we also need to exclude 4 Royal Flushes and 36 Straight Flushes, so the total is $4\binom{13}{5} - 40$.

Straight: Five cards in order, possibly of different suits (Example: 5♥, 6♠, 7♦, 8♠, 9♣). Which card to start from (anything from A to 10) 10×, five suits $4^5 = 10 \cdot 4^5$. From here, we also need to exclude Royal Flushes and Straight Flushes, so the final answer is $10 \cdot 4^5 - 40$.

Triple: Three cards of the same value, and two other random cards (Example: K♥, K♠, K♦, 4♠, 2♣). Which card $\binom{13}{1}\times$, which three suits $\binom{4}{3}\times$, which two other values $\binom{12}{2}\times$, which two suits for these two random cards 4^2 , total $= \binom{13}{1}\binom{4}{3}\binom{12}{2}4^2$.

Two pairs: Two cards of the same value, two cards of the same value, and a random card (Example: K♥, K♠, 10♦, 10♠, 4♣). Which two cards $\binom{13}{2}\times$, two suits for each of pair $\binom{4}{2}^2\times$, remaining value 11×, remaining suit 4, total $= \binom{13}{2}\binom{4}{2}^2 \cdot 11 \cdot 4$.

Pair: Two cards of the same value, and three other random cards (Example: K♥, K♠, Q♦, 4♠, 2♣). Which card $\binom{13}{1}\times$, which two suits $\binom{4}{2}\times$, which three other values $\binom{12}{3}\times$, which three suits for these three random cards 4^3, total $=\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3$.

To calculate probabilities of each of these combinations, we have to divide the counts above by the total number of poker hands, which is $\binom{52}{5}$. The table below gives the probabilities and odds.

Combination	Count	Probability	Odds
Royal Flush	4	0.000154%	1 : 649,740
Straight Flush	36	0.00139%	1 : 72,192
Four of a Kind	$13 \cdot 12 \cdot 4$	0.024%	1 : 4,165
Full House	$13\binom{4}{3} \cdot 12\binom{4}{2}$	0.1441%	1 : 693
Flush	$4\binom{13}{5} - 40$	0.1965%	1 : 508
Straight	$10 \cdot 4^5 - 40$	0.3925%	1 : 254
Triple	$\binom{13}{1}\binom{4}{3}\binom{12}{2}4^2$	2.1128%	1 : 46.3
Two Pairs	$\binom{13}{2}\binom{4}{2}^2 \cdot 11 \cdot 4$	4.7539%	1 : 20
Pair	$\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3$	42.2569%	1 : 1.37
Nothing		50.1177%	1 : 0.995

Homework assignment

In all the problems, you can write your answer as a combination of factorials and $\binom{n}{k} = {}_nC_k$ – you do not have to do the computations. And, as usual, please write your reasoning, not just the answers! Some problems are repeated from previous homeworks - may skip the problems you have already solved.

1. In poker, players are drawing "hands" (combinations of 5 cards) from the 52-card deck (4 suits, 13 cards in each).
    a. How many possible hands are there?
    b. How many hands in which all cards are spades?
    c. What are your chances of drawing a hand in which all cards are spades?
    d. What are your chances of drawing a hand which has 4 queens in it? [Hint: how many such hands are there?]
    e. What are your chances of drawing a hand which has exactly 3 queens in it?
    f. What are your chances of drawing a royal flush (Ace, King, Queen, Jack, 10 — all of the same suit)? [Hint: what are your chances of drawing a royal flush in a given suit, say spades?]
2. A senior class in a high school, consisting of 120 students, wants to choose a class president, vice president, and 3 steering committee members. How many ways are there for them to do this?

3. How many five letter “words” are there such that the letters are in alphabetical order? (Here a “word” is any sequence of letters from the alphabet a through z.)
4. Andrew has 7 pieces of candy, and Tim has 9 (all different). They want to trade 5 pieces of candy. How many ways are there for them to do it?
5. In one of the lotteries run by New York State, “Sweet Million”, they randomly choose 6 numbers out of numbers 1–40. If you guess all 6 correctly (order does not matter), you win $1,000,000. [There are also smaller prizes for guessing 5 out of 6, etc., but let us ignore them for now.]
    a. How many ways are there to choose 6 numbers out of 40?
    b. What are your chances of winning?
    c. If a lottery ticket cost $1, how much money does New York State make for each ticket sold (on average)?
    d. *If you choose 6 numbers out of 40 at random, what are the chances that exactly 5 of them will be winning numbers?

    Bonus question: find online the rules for another NY lottery, “Mega Millions”, and analyze your chances to win.
6. We toss a coin 100 times.
    a. What is the probability of obtaining all tails? exactly 2 heads? exactly 50 heads? at least 1 head?
    b. Same question for an unfair coin, which gives heads with probability $p = 0.45$ and tails with probability $q = 0.55$.
7. A monomial is a product of powers of variables, i.e. an expression like x^3y^7.
    a. How many monomials in variables x, y of total degree of exactly 15 are there? (Note: this includes monomials which only use one of the letters, e.g. x^{15}.)
    b. Same question about monomials in variables x, y, z. [Hint: if you write 15 letters in a row, you need to indicate where x’s end and y’s begin — you can insert some kind of marker to indicate where it happens.]
    c. How many monomials in variables x, y of degree at most 15 are there?
    d. *How many monomials in variables x, y, z of degree at most 15 are there?
8. A frog on vacation to India is attempting to climb a step at the Taj Mahal. The frog, figuring it has a one in ten chance to succeed at the jump, decides to attempt the jump ten times (if it succeeds early, it won’t make further attempts). What is the chance that the frog will make it up the step after its ten attempts?
9. Three rooks are said to be friendly if they are in three distinct but consecutive rows on the chessboard. How many ways are there to put three rooks on a chessboard so that they are friendly? (Assume the rooks are all the same color.)
10. If you have 5 lines on the plane so that no two are parallel and there are no triple intersection points, how many triangles do they form? What if there are n lines?

Recap: main formulas of Combinatorics

- The **number of permutations**, or the number of ways to order n items is,

$$n! = n(n-1)(n-2) \dots \cdot 2 \cdot 1$$

- The number of ways to choose k items out of n **if the order matters**:

$$_nP_k = \frac{n!}{(n-k)!} = n(n-1)(n-2) \dots \cdot (n-k+1)$$

- The number of ways to choose k items out of n **if the order does not matter**:

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2) \dots \cdot (n-k+1)}{k(k-1)(k-2) \dots \cdot 2 \cdot 1}$$

The rationale behind this formula is simple. To find all possible choices of how to fill the k out of n positions with k out of n items, we first count all permutations of the given n items, which is $n!$. This counts all possible choices of k items out of n. However, the same choice is counted multiple times: for each possible choice of k items, any permutation of the remaining $n-k$ items is counted among the $n!$ permutations. Hence, we need to divide $n!$ with $(n-k)!$ to only count different choices of k items. If permutations among the k items do not matter, we also need to divide by $k!$

Combinatorics with repetitions

This type of problem is equivalent to placing $k = 10$ identical marbles into $n = 5$ different jars. A common way to illustrate the solution is the "stars and bars" diagram:

1. First, place all 10 marbles ("stars") in a row; they are all the same, so the order does not matter.
2. Then, divide this row into $n = 5$ groups by inserting $(5-1) = 4$ dividers ("bars") anywhere in the row; the "bars" are also identical.
3. Counting from the left, each group of stars is the number of marbles for the jar; if two "bars" are next to each other, then the corresponding jar will be empty.

Below is an illustration for $k = 10$ and $n = 5$:

$$\overbrace{**********}^{n\ stars} = \overbrace{*\ |\ ***\ |\ **\ ||\ ****}^{(n+k-1) stars and\ bars}$$

All possible permutations of n identical "stars" and $(k-1)$ "bars" give you all possible ways to split into k groups (jars). We counted such permutations in the previous class:

$$(Permutations\ of\ n\ stars, k-1\ bars) = \frac{(k+n-1)!}{n!\,(k-1)!} = \binom{k+n-1}{k-1} = \binom{k+n-1}{n}$$