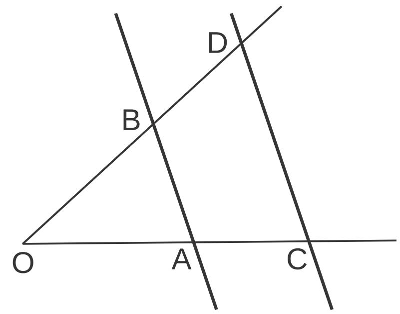
February 9, 2025 Math 8

Handout 20. Euclidean Geometry 9: Thales's theorem. Similar triangles.

Thales's theorem.

Theorem 40 (Thales's theorem)**.** Let lines AB and CD intersect the sides OA and OB of angle $\angle AOB$ at points A, C, and B, D, respectively, as shown in the picture. Then, lines AB and CD are parallel if and only if,

$$\frac{|OA|}{|OB|} = \frac{|OC|}{|OD|}$$

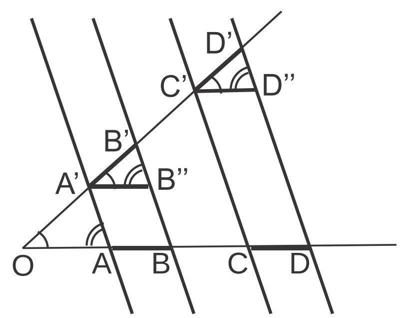


Or equivalently,

$$\frac{|OA|}{|OC|} = \frac{|OB|}{|OD|}$$

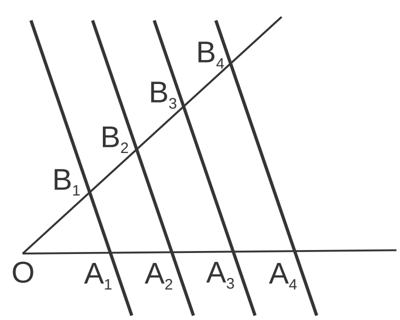
Exercise. Prove that in this case also $\frac{|OA|}{|OB|} = \frac{|AC|}{|BD|}$.

Proof. We have already considered and proved a special case of this theorem when discussing the midline of a triangle. In general case the proof of this theorem is (un)expectedly difficult (why do you think?). In the special case when $\frac{|OA|}{|OC|}$ is a rational number, the theorem follows directly from a simpler theorem which we prove below using the properties of a parallelogram, similar to how we proved properties of the midline. The case of irrational numbers is hard and requires advanced reasoning similar to taking a limit. We skip the full proof for now; it will be discussed in Math 9. ☐



Theorem 41. Let segments $\overline{AB}$ and $\overline{CD}$ on one side of an angle $\angle AOA'$ be congruent, $\overline{AB} \cong \overline{CD}$, and let parallel lines AA', BB', CC', and DD' intersect the other side at the points A', B', C', and D', respectively, as shown in the figure. Then segments $\overline{A'B'}$ and $\overline{C'D'}$ are also congruent $\overline{A'B'} \cong \overline{C'D'}$.

Proof. Left as a homework exercise (consider the figure). ☐



Corollary. If on one side of an angle $\angle AOB$ we mark consecutive congruent segments, $\overline{A_1A_2} \cong \overline{A_2A_3} \cong \overline{A_3A_4} \cong \cdots$, and draw parallel lines, $A_1B_1 \parallel A_2B_2 \parallel A_3B_3 \parallel A_4B_4 \ldots$, then segments cut by these lines on the other side of the angle are also congruent, $\overline{B_1B_2} \cong \overline{B_2B_3} \cong \overline{B_3B_4} \cong \cdots$.

Exercise. Using the above theorem, prove Thales's theorem for the case when $\frac{|OA|}{|OC|} = \frac{m}{n}$ is a rational number.

Similar triangles.

Definition. If sides of triangles $\triangle ABC$ and $\triangle A'B'C'$ belong to respectively parallel lines, $AB \parallel A'B'$, $BC \parallel B'C'$, $CA \parallel C'A'$, then the two triangles are similar, denoted as $\triangle ABC \sim \triangle A'B'C'$. Symmetry operations and rotations preserve the similarity relation between triangles. Namely, if $\triangle ABC \sim \triangle A'B'C'$ and $\triangle A''B''C''$ is related to $\triangle A'B'C'$ via rotation (including central symmetry), axial symmetry, or a combination of these, then $\triangle A''B''C'' \sim \triangle ABC$.

Another way to state the definition is,

Definition. If using symmetry transformations and rotations triangle $\triangle A'B'C'$ can be positioned such that its sides are respectively parallel to the sides of $\triangle ABC$, $AB \parallel A'B'$, $BC \parallel B'C'$, $CA \parallel C'A'$, then the two triangles are similar, $\triangle ABC \sim \triangle A'B'C'$.

Theorem 42 (AA(A) similarity test)**.** If the corresponding angles of triangles $\triangle ABC$ and $\triangle A'B'C'$ are equal, $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$, then the triangles are similar, $\triangle ABC \sim \triangle A'B'C'$. Note that we need to compare only two pairs of angles, and then the third pair will be also equal.

Theorem 43 (SSS similarity test)**.** If the corresponding sides of triangles $\triangle ABC$ and $\triangle A'B'C'$ are proportional:

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|AC|}{|A'C'|}$$

then the triangles are similar, $\triangle ABC \sim \triangle A'B'C'$.

The common ratio, $\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|AC|}{|A'C'|} = k$, is sometimes called the similarity coefficient.

Theorem 44 (SAS similarity test). If two pairs of corresponding sides of triangles $\triangle ABC$ and $\triangle A'B'C'$ are proportional:

$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|}$$

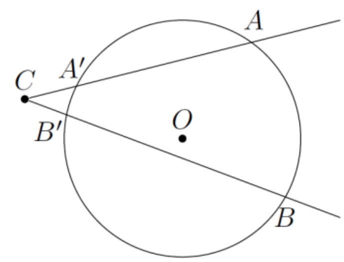
and angles between these sides are equal, $\angle A \cong \angle A'$, then the triangles are similar, $\triangle ABC \sim \triangle A'B'C'$.

Proofs of all of these similarity test theorems follow directly from Thales theorem.

Homework problems

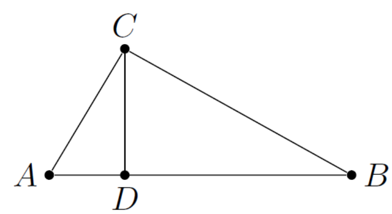
This homework may be more challenging than usual. Try to solve as many problems as you can, and we will discuss them all in class.

1. (A modification of Inscribed Angle Theorem.) Consider a circle and an angle whose vertex C is outside this circle and both sides intersect this circle at two points as shown in the figure. In this case, the intersection of the angle with the circle defines two arcs: $\widehat{AB}$ and $\widehat{A'B'}$. Prove that in this case $m\angle C = \frac{1}{2}\left(\widehat{AB} - \widehat{A'B'}\right)$.

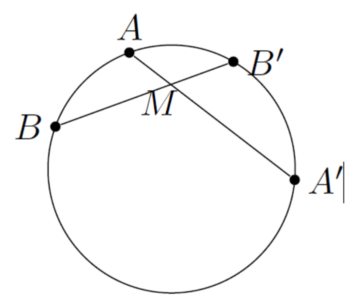


[Hint: first draw the line AB' and find the angle $\angle AB'B$. Then notice that this angle is an exterior angle of $\triangle ACB'$.]

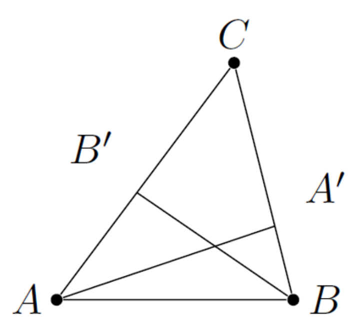
2. Can you suggest and prove an analog of the previous problem, but when the point C is inside the circle (you will need to replace an angle by two intersecting lines, forming a pair of vertical angles)?



3. Prove Corollary to Theorem 41 using Thales Theorem. Hint: let $\frac{|OB_1|}{|OA_1|} = k$ and show that then $|B_iB_{i+1}| = k|A_iA_{i+1}|$.
4. Using Theorem 41 and its corollary, describe how one can divide a given segment into 5 equal parts using ruler and compass.



5. Given segments of length a, b, c, construct a segment of length $\frac{ab}{c}$ using ruler and compass.
6. Let $\triangle ABC$ be a right triangle with $\angle C = 90°$, and let CD be the altitude. Prove that triangles $\triangle ACD$ and $\triangle CBD$ are similar. Deduce from this that $|CD|^2 = |AD| \cdot |DB|$.
7. Let M be a point inside a circle and let AA', BB' be two chords through M. Show that then $|AM| \cdot |MA'| = |BM| \cdot |MB'|$. [Hint: use inscribed angle theorem to show that $\triangle AMB \sim \triangle B'MA'$].



8. Let AA' and BB' be altitudes in the acute triangle $\triangle ABC$.
    a. Show that points A', B' are on a circle with diameter AB.
    b. Show that $\angle AA'B' \cong \angle ABB'$, $\angle A'B'B \cong \angle A'AB$
    c. Show that triangle $\triangle ABC$ is similar to triangle $\triangle A'B'C$.
9. (Chords intersecting outside the circle). Consider circle $\omega(O, R)$, its chord $\overline{AA'}$, a point C on the line (AA') outside the circle, and the tangent $\overline{CD}$ to the circle. Using similar triangles, prove that
    a. $|CA| \cdot |CA'| = |CD|^2$.
    b. for any chords $\overline{AA'}$, $\overline{BB'}$ intersecting at point C outside the circle, $|CA| \cdot |CA'| = |CB| \cdot |CB'|$. [Hint: connect point A to D and consider inscribed and tangent-chord angles.]