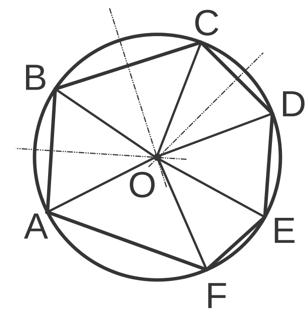
February 2, 2025 Math 8

Handout 19. Euclidean Geometry 8: Inscribed and circumscribed polygons. Angles in polygons. Axial symmetry. Central symmetry. Rotation.

Inscribed and circumscribed polygons.

Definition. If all vertices of a polygon lie on a circle, then the polygon is called inscribed into the circle and the circle is called circumscribed around the polygon, or circumcircle. The center of the circumcircle is called the circumcenter. Not every polygon has a circumcircle: for example, a concave polygon does not. A polygon inscribed in a circle is said to be a cyclic polygon (see polygon $ABCDEF$ in the figure).



Exercise. Prove the following theorems, which we have previously discussed.

a. About any triangle, a circle can be circumscribed, and such a circle is unique.
b. Into any triangle a circle can be inscribed, and such a circle is unique.

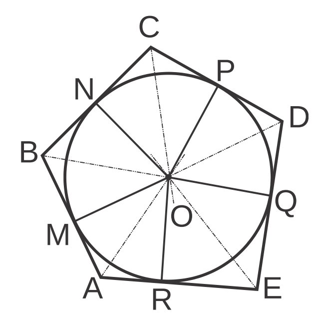
Exercise. Prove that the circumcenter lies inside a triangle if and only if the triangle is acute, lies outside a triangle if and only if the triangle is obtuse, and lies at the midpoint of the hypotenuse for a right triangle.

Last time, we have also proven the following theorem providing necessary and sufficient condition for a quadrilateral to be cyclic.

Theorem 27. A quadrilateral $ABCD$ is cyclic (can be circumscribed by a circle) if and only if the sum of two opposite angles is equal to 180°: $\angle ABC + \angle ADC = 180° = \angle BAD + \angle BCD$.

There is no extension of this sort of condition for a general polygon. The following theorem provides a necessary and sufficient condition for a general polygon (including triangles and quadrilaterals considered before) to be cyclic.

Theorem 32. A polygon is cyclic (has a circumscribed circle) if and only if bisectors (mid-perpendiculars) of all its sides are concurrent and meet in a single point, the circumcenter O. More generally, for n distinct points there are $n(n-1)/2$ bisectors, and the concyclic condition is that they all meet in a single point, the center O. All concyclic points are equidistant from the center of the circle.



Proof. Left as a homework exercise. □

Connecting vertices of an inscribed polygon with the circumcenter O, subdivides the polygon into an isosceles triangles sharing the vertex O.

Definition. A circle contained inside of a polygon and tangent to each of the polygon's sides is called an inscribed circle, or incircle, and its center is called the incenter of the polygon. Not every polygon has an incircle. A convex polygon that contains an inscribed circle is called a circumscribed polygon, or a tangential polygon ($ABCDE$ in the figure).

A cyclic polygon with the vertices at the tangency points ($MNPQR$), which has the inscribed circle of the tangential polygon passing through each of its vertices is the dual polygon of the tangential polygon. The following theorems provide a necessary and sufficient condition for a polygon to be tangential.

Theorem 33. A convex polygon has an incircle if and only if all of its internal angle bisectors are concurrent. Their common point O is the incenter (the center of the incircle).

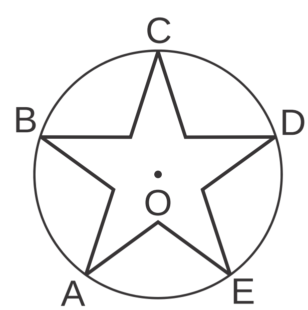
Proof. Left as a homework exercise. □

Theorem 34. A polygon with n sequential sides $a_1, \dots, a_n$ is tangential if and only if the system of equations $x_1 + x_2 = a_1,\ x_2 + x_3 = a_2, \dots, x_n + x_1 = a_n$ has a solution $(x_1, \dots, x_n)$ in positive real numbers. If such a solution exists, then $x_1, \dots, x_n$ are the tangent lengths of the polygon (the lengths from the vertices to the points where the incircle is tangent to the sides).

Proof. Left as a homework exercise. □

If the number of sides n is odd, then for any given set of side lengths satisfying the existence criterion of Theorem 34 there is only one tangential polygon. But if n is even there are an infinitude of them. For example, in the quadrilateral case where all sides are equal, we can have a rhombus with any value of the acute angles, which are all tangential to an incircle.

Last time, we have proven that the area of a polygon with inscribed circle of radius r and perimeter P is $S = \frac{1}{2}Pr$.

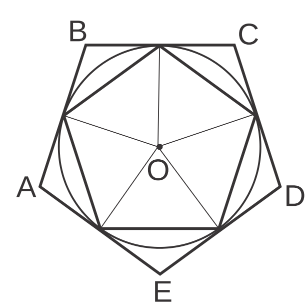


Definition. A regular polygon is a polygon that has angles equal in measure and equilateral (all sides have the same length).

Regular polygons may be either convex or star. If the perimeter or area is fixed, a sequence of regular polygons with an increasing number of sides in the limit approximates a circle.

Theorem 35. A regular polygon is a cyclic polygon, i.e., all vertices of a regular polygon lie on a common circle (the circumscribed circle).

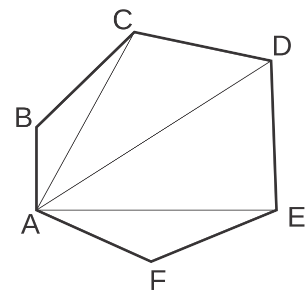
Proof. Left as a homework exercise. □



Theorem 36. A regular (convex) polygon also has an incircle that is tangent to every side at the midpoint. Thus, a regular convex polygon is a tangential polygon.

Proof. Left as a homework exercise. □

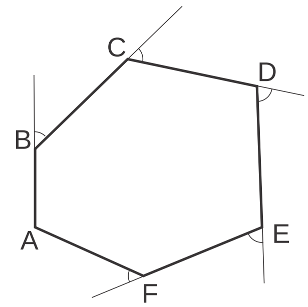
Angles in a polygon.



Theorem 37. The sum of angles of a convex polygon with n sides is $180° \times (n-2)$.

Proof. Draw diagonals dividing the polygon into $(n-2)$ triangles as shown in the Figure and then use the corollary of the parallel axiom that sum of angles of a triangle is 180°. Alternatively, take a point inside the polygon and divide it into n triangles by connecting the point with the vertices. ◻

Exercise. Prove that the same is true for a non-convex polygon.

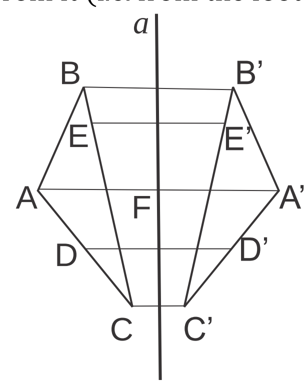


Theorem 38. The sum of the exterior angles of a convex polygon formed by consecutively extending one of its sides at each vertex (see Figure) is 360°.

Proof. Each of the obtained exterior angles supplements the adjacent interior angle to 180°. Adding them, we obtain $180° \times n - 180° \times (n-2) = 360°$. ◻

Axial symmetry.

Definition. Two points, A and A', are symmetric with respect to a line a if they are located on opposite sides of this line on a perpendicular to line a at the same distance from it (i.e. from the foot F of the perpendicular $\overline{AA'}$, $|FA| = |FA'|$).



Definition. Two figures (or two parts of the same figure) are called symmetric about a line if for each point of one figure (A, B, C, D, E,… in the Figure) the point symmetric to it with respect to this line (A', B', C', D', E',… in the Figure) belongs to the other figure, and vice versa.

Two symmetric figures can be superimposed by folding the plane about the axis of symmetry, i.e., by rotating one figure out-of-plane about an axis of symmetry by 180°. However, symmetric figures cannot be superimposed if they remain in the plane (i.e. via translations or rotations).

Definition. A figure is said to have an axis of symmetry a if this figure is symmetric to itself about line a, i.e., if for any point of the figure its symmetric point also belongs to the figure. In this case, the axis of symmetry a divides the figure into two symmetric parts.

Exercise. Which parallelograms possess an axis of symmetry?

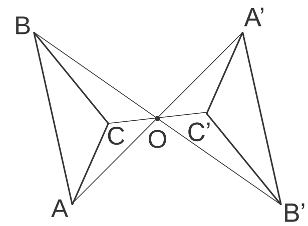
Theorem 36. The bisector of the angle at the vertex of an isosceles triangle is an axis of symmetry of the triangle.

Proof. Left as a homework exercise. ◻

Axial symmetry is often encountered in nature, art, and science (e.g., a butterfly, a leave, an orbital).

Central symmetry.

Definition. Two points, A and A', are symmetric about a point O if O is a midpoint of the line segment $\overline{AA'}$, $|AO| = |OA'|$.

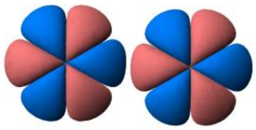


To find a point symmetric to A about O one extends segment AO past O and finds point A' on the ray AO such that $\overline{AO} \cong \overline{A'O}$.

Definition. Two figures (or two parts of the same figure) are called symmetric about a given point O, if for each point of one figure (A, B, C,… in the Figure) the point symmetric to it about the point O (A', B', C',… in the Figure) belongs to the other figure, and vice versa. The point O is called the center of symmetry.

Definition. A figure is said to have a center of symmetry O if this figure is central-symmetric to itself about center O, i.e., if for each point of the figure its symmetric point about center O also belongs to the figure. The center of symmetry lies inside a figure; an example of a central-symmetric figure is a circle, where its center is the center of symmetry.

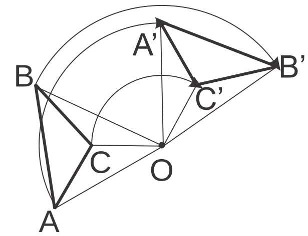
Exercise. Prove that in a parallelogram, the intersection point of the diagonals is the center of symmetry.



Like axial symmetry, central symmetry is often encountered in nature, art, and science. What symmetries are present in the above figures?

Rotations about a point.

Definition. For any point A on the plane, the point A' obtained by a rotation by an angle α about a point O is the endpoint of an arc of the circle with the center O and radius $r = |OA|$, which begins at point A and spans angle α.



Definition. A figure obtained from a given figure by a rotation around point O by an angle α consists of all points (A', B', C',… in the Figure) obtained by this rotation from each point of the given figure, (A, B, C,… in the Figure).

One can think of a rotation as a transformation of a figure achieved by moving the figure on the plane where each point of the figure is rigidly located at the end of the fixed-length segment connecting it to the center of rotation and angles between all such segments are fixed, so that they all rotate together.

Two symmetric figures can be superimposed by rotating one figure about the center of symmetry by 180°. Indeed, this is true for any two symmetric points and therefore for all points in the figures. Hence, unlike the case of axial symmetry, two central-symmetric figures can be superimposed while remaining within the plane.

Homework problems

Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

1. How many axes of symmetry does have
   a. an equilateral triangle?
   b. an isosceles triangle which is not equilateral?
2. How many axes of symmetry can a quadrilateral have?
3. A kite is a quadrilateral symmetric about its diagonal. Give an example of (i) a kite; (ii) a quadrilateral which is not a kite but has an axis of symmetry. Prove that
   a. A quadrilateral is a kite if it has an axis of symmetry passing through a vertex
   b. Diagonals of a kite are perpendicular
4. Can a pentagon have an axis of symmetry passing through two (one, none) of its vertices?
5. Can a polygon have two parallel axes of symmetry?
6. How many axes of symmetry does a regular n-gon (regular polygon with n sides) have
   a. If n is even?
   b. If n is odd?
7. *Points A and B lie on the same side of a line MN. Find point C on MN such that line MN would make congruent angles with the sides of the broken line ACB.
8. How many centers of symmetry can a polygon have?
9. Prove that the line segment connecting any point on one base of a trapezoid with any point on the other base is bisected by the midline of the trapezoid.
10. Prove that in a right triangle the median to the hypotenuse is congruent to the half of it.
11. Find the geometric locus of the midpoints of all segments drawn from a given point to various points on a given line.