September 15, 2024 Math 8

Baseline revue test. Algebra.

1. In the first quarter of the year, the retirement fund of Alex's parents lost 15% of its value, in the second quarter it lost another 5%. Then, in the third and fourth quarters it gained 10% and 20%, respectively.
    a. What is the total net gain/loss of Alex's parents' retirement fund over the year (in percent)?
    b. What is the mean quarterly gain/loss over the year?
2. Open parentheses and expand the following expressions.
    a. $(a+b)^2 =$
    b. $(a-b)^2 =$
3. Factor the following expressions
    a. $a^2 - b^2 =$
    b. $a^2 + b^2 =$
4. For a quadratic equation $ax^2 + bx + c = 0$ the roots are,
   $x_{1,2} =$
   and they have the following properties,
   $x_1 + x_2 =$
   $x_1 \cdot x_2 =$
5. What is the number of permutations of n objects?
6. How many ways are there to select k objects out of n if,
    a. order does matter?
    b. order does not matter?
7. Write the formula for a binomial coefficient

$$C_n^k \equiv {}_nC_k \equiv \binom{n}{k} =$$

   and explain its relation to combinatorics and certain counting problems.

8. How many different prime factors does 2024^{2024} have? How many prime factors does it have in total (that is, when decomposed into a product of primes, how many prime factors, some of them being equal, are there in that product)?

Math 8 placement test 2024

1. Write the following number in the form $a + b\sqrt{3}$, with rational a, b,
$$\frac{\sqrt{3}}{4+\sqrt{3}} = ?$$
2. A certain society needs to elect a committee consisting of a chairman and 3 members. How many ways are there for them to do that if there are 12 candidates?
3. 6 kids are choosing toys from a toy bin. There are 10 toys in the bin, all different. How many possible choices are there?
4. Factor the following expression:
$$x^4 - 9y^4$$
5. Solve the equation:
$$(2x-1)(x+1) = 9$$
6. Solve the inequality:
$$\frac{x+3}{x-1} \geq 0$$
7. A right triangle with angles 90, 60, and 30 degrees is inscribed in a circle with diameter 2. Find the area of this triangle.

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Handout 1: Combinatorics review

Main formulas of Combinatorics.

- The number of ways to order n items is,

$$n! = n(n-1)(n-2)\dots\cdot 2\cdot 1$$

- The number of ways to choose k items out of n **if the order matters**:

$${}_nP_k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\dots\cdot(n-k+1)$$

- The number of ways to choose k items out of n **if the order matters**:

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\dots\cdot(n-k+1)}{k(k-1)(k-2)\dots\cdot 2\cdot 1}$$

These numbers are the ones that appear in Pascal triangle and in many other problems:

$\binom{n}{k} = {}_nC_k$ = The number of paths on the chessboard going k units up and $n-k$ to the right

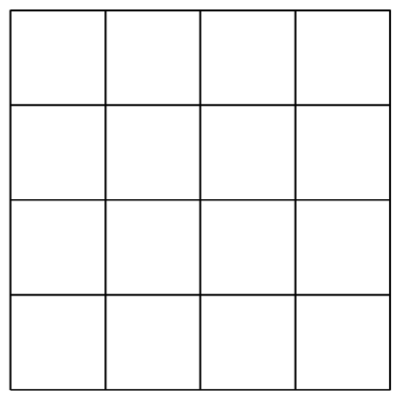
= The number of words that can be written using k zeros and $n-k$ ones

= The number of ways to choose k items out of n if the order does not matter

Note: when solving combinatorial problems, you don't always need to compute the actual numbers, because they can be HUGE! For example, in problem 8 blow, it's better to keep 60! ("sixty-factorial") in your solution as-is, because the actual number is larger than 10^{80}.

Combinatorics review problems (homework assignment).

1. A club consisting of 25 people need to choose the president, vice-president, and treasurer. In how many ways can they do this?
2. In a meeting of 25 people, every one of them shakes hands once with every other. How many handshakes was it altogether?
3. There is a round table seating 8. How many ways there are for 8 people to choose their seats at the table? What if we do not distinguish between two seatings which only differ by rotating the table?
4. How many words one can get by permuting letters of the word “tiger”? of the word “rabbit”? of the word “common”? of the word “Mississippi”?
5. If we draw 3 cards out of the deck of 52 cards (4 suits   13 values), what are the chances that
    a. They will all be all spades
    b. They will be all aces
    c. That they will be ace of spades, queen of spades, and king of spades, in this order
    d. That they will be queen of spades, ace of spades, and king of spades, in this order
    e. * That they will be ace, queen, and king of spades, in some order
6. How many different paths are there on 4×4 chessboard connecting the lower left corner with the upper right corner? What about 5×5? The path should always be going to the right or up, never to the left or down.



7. How many “words” of length 5 one can write using only letters U and R, namely 3 Us and 2 Rs? What if you have 5 Us and 3 Rs? [Hint: it is related to the previous problem – each such “word” can describe a path on the chessboard, U for up and R for right…]
8. A drunkard is walking along a road from the pub to his house, which is located 1 mile north of the pub. Every step he makes can be either to the north, taking him closer to home, or to the south, back to the pub – and it is completely random: every step with can be north of south, with equal chances. What is the probability that after 60 steps, he will end up
    a. at the starting position
    b. 2 steps north from the starting position
    c. 1 step north from the starting position
    d. 10 steps north from the starting position
    e. 8 steps north from the starting position
9. * You have 10 books which you want to put on 2 bookshelves. How many ways are there to do it (order on each bookshelf matters)?

<u>Solutions to the baseline revue test. Algebra.</u>

1. In the first quarter of the year, the retirement fund of Alex's parents lost 15% of its value, in the second quarter it lost another 5%. Then, in the third and fourth quarters it gained 10%and 20%, respectively.
   a. The total net gain/loss of Alex's parents' retirement fund over the year is a factor of, $y = 0.85 \cdot 0.95 \cdot 1.1 \cdot 1.2 = 1.0659$, or 6.59%
   b. The mean quarterly gain, x, is a factor, which being equally applied for all quarters yields the same yearly gain, $x \cdot x \cdot x \cdot x = x^4 = 1.0659$, so $x = \sqrt[4]{1.0659} \approx 1.016$, or 1.6%.
2. Open brackets and expand the following expressions
   a. $(a+b)^2 = a^2 + 2ab + b^2$
   b. $(a-b)^2 = a^2 - 2ab + b^2$
3. Factor the following expressions
   a. $a^2 - b^2 = (a-b)(a+b)$
   b. $a^2 + b^2 = \ldots$
4. For a quadratic equation $ax^2 + bx + c = 0$ the roots are,
   $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
   and they have the following properties,
   $x_1 + x_2 = -\frac{b}{a}$
   $x_1 \cdot x_2 = \frac{c}{a}$

Although this can be checked by direct substitution of the formula for $x_{1,2}$, the easiest way to see this is by rewriting the equation in the reduced form and identifying it with the product of the two brackets,

$$ax^2 + bx + c = 0 \Leftrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \Leftrightarrow (x - x_1)(x - x_2) = 0 \Leftrightarrow x^2 - (x_1 + x_2)x + x_1 x_2 = 0$$

This is an example of the polynomial factorization, which we will be studying in significant detail this year.

5. What is the number of permutations of n objects? Answer: $n!$

This is the number of ways that n different objects (or subjects) can be placed into n different places.

Examples:

- How many ways is there to sit n people in a movie theater with n numbered chairs?
- How many ways is there to hand out n different books to n students?
- How many ways is there to place n numbered billiard balls into n numbered spots?

There is n ways to select a place for the first object (subject), for each of these n choices there is $n-1$ choice to place the second one, so there are $n(n-1)$ in total different choices to fill the first two spots, and so on. Hence, there are $n! = n(n-1)(n-2) \ldots \cdot 2 \cdot 1$.

6. How many ways is there to select k objects out of n if,
    a. order does matter? Answer: ${}_nP_k = \frac{n!}{(n-k)!}$
    b. order does not matter? Answer: ${}_nC_k = \frac{n!}{k!(n-k)!}$
7. How many different prime factors does 2024^{2024} have? How many prime factors does it have in total (that is, when decomposed into a product of primes, how many prime factors, some of them being equal, are there in that product)?

$2024^{2024} = (8 \cdot 11 \cdot 23)^{2024} = (2^3 \cdot 11 \cdot 23)^{2024} = 2^{3 \cdot 2024} \cdot 11^{2024} \cdot 23^{2024}$. Hence, 2024^{2024} has 3 different prime factors, 2, 11, and 23, and its prime decomposition is a product of $5 \cdot 2024 = 10120$ prime factors.