May 11, 2025

Math battle 8.

1. There are 365 students in the graduating class of 2025 in the Eric Blair High School who were all born in 2007. What is the probability that none of the students has birthday today?
2. Find all integer numbers which give remainder 2 upon division by 7 and remainder 5 upon division by 13.
3. Is the product, $(2025 \cdot 2026 \cdot 2027 \cdot \ldots \cdot 4048)$, divisible by 2024! (2024 factorial)? What about $(2024 \cdot 2025 \cdot 2026 \cdot \ldots \cdot 4047)$?
4. You have 100 beads, all different. How many different necklaces of length 50 can you make using these beads? The clasp on the necklace can be ignored, in the sense that 1-2-clasp-3-4 is the same as 1-clasp-2-3-4.
5. 10 students are solving problems of a math competition. Each of 10 problems was solved by the same number of students, but no two students have solved the same number of problems. One of the students, Max, has solved problems 1 through 5 and has not solved problems 6 through 9. Has Max solved problem 10?
6. Given an equilateral triangle ABC, find all points M on the plane such that both triangles ABM and ACM are isosceles.

Bonus problems.

7. 20 chess players meet for a championship. On the first day, each player played one game. On the second day, each player also played one game (possibly with the same player (s)he played the first day). Is it true that after these two days it is always possible to choose 10 players so that no two of them played against each other?
8. Let us call a number between 0 and 999999 "lucky" if the sum of the first 3 digits is equal to the sum of the last 3 digits (we write all numbers as 6-digit numbers, adding zeros in front, if necessary, e.g., writing 17 as 000017). Is the total number of lucky numbers even or odd?
9. Alice and Bob are playing the following game. They have a staircase with 1001 steps; on some of these steps, there are stones (no more than one stone on each step). At her turn, Alice can take any stone and move it to the nearest free step above it. After that, Bob can take any stone such that the step immediately below is empty and roll this stone one step down. Initially there are 500 stones occupying steps 1-=500. Alice goes first; her goal is to get a stone to step 1001. Bob's goal is to prevent Alice from achieving this. Is there a winning strategy for one of them? If so, what is this strategy?
10. Is it possible to put in each vertex of any given triangle ABC a number such that for every edge, its length is equal to the sum of numbers at its endpoints?
11. Let ABC be a right triangle, with $\angle A = 90°$, and $K \in BC$ be such that $AB = AK$. If we know that segment AK bisects the angle bisector CL, what are the angles of ΔABC?
12. Given a circle and a point P inside this circle, construct a chord AB through P such that $|AP| - |BP| = 2\text{cm}$.

Extra problems.

1. How many natural numbers <1000 are not divisible by 7, 9 and 13?
2. Four friends, A., B., C. and D., decided to exchange presents. They agreed that each one prepares a present, which will then be randomly drawn. Hence, each can get, with equal probability, any of the four presents. What is the probability that no one gets his/her own present, while A. gets the present from D.?
3. A city has 10 bus routes. Is it possible to arrange the routes and the bus stops so that if one route is closed, it is still possible to get from any one stop to any other (possibly changing the route along the way), but if any two routes are closed, there are at least two stops such that it is impossible to get from one to the other?
4. In the number 454**, find the missing digits so that the number is divisible by 2, by 7, and by 9.
5. Prove that the product of any m consecutive integer numbers is divisible by $m!$