

MATH 7: HANDOUT 18
MORE INEQUALITIES. SNAKE METHOD.

SOLVING POLYNOMIAL INEQUALITIES

In addition to linear inequalities, we can also consider polynomial inequalities: they would have terms like x^2, x^3 , etc. The general rule for solving polynomial inequalities is as follows:

- Find the roots and factor your polynomial, writing it in the form $p(x) = a(x - x_1)(x - x_2)$ (for polynomial of degree more than 2, you would have more factors).
- Roots x_1, x_2, \dots divide the real line into intervals; define the sign of each factor and the product on each of the sign intervals.
- If the inequality has \geq or \leq signs you should also include the roots themselves into the intervals.

Example 1. $x^2 + x - 2 > 0$.

Solution. We find roots of the equation $x^2 + x - 2 = 0$ and obtain $x = -2, 1$. The inequality becomes $(x + 2)(x - 1) > 0$ and roots $-2, 1$ divide the real line into three intervals $(-\infty, -2), (-2, 1), (1, +\infty)$. It is easy to see that the polynomial $x^2 + x - 2$ is positive on the first and the third intervals and negative on the second one. The solution of the inequality is then $x < -2$ or $x > 1$. We sometimes, write this also as $x \in (-\infty, -2) \cup (1, +\infty)$. (sign \cup means “or”).

Example 2. $-x^2 - x + 2 \geq 0$.

Solution. We have $-(x + 2)(x - 1) \geq 0$. The left hand side is positive for $-2 < x < 1$. As the sign in the inequality is \geq we have to include the roots into the interval and obtain $-2 \leq x \leq 1$. One can also write $x \in [-2, 1]$ (square brackets here mean that the endpoints of the interval are included).

Example 3. $x^2 + x + 2 \geq 0$.

Solution. The polynomial here does not have roots (the discriminant $1^2 - 4 \cdot 1 \cdot 2 < 0$). Therefore, the real line is not divided into the intervals, which means that the polynomial is of the same sign for all x . We check that it is positive, for example, for $x = 0$. The solution is that x is any number. We can write $x \in (-\infty, +\infty)$.

Example 4. $x^2 + x + 2 < 0$.

Solution. The polynomial does not have roots and is positive everywhere. This means that the inequality does not have solutions at all. One can also write $x \in \emptyset$.

Example 5. $x^2 - 2x + 1 > 0$.

Solution. The inequality is $(x - 1)^2 > 0$. There is only one root here which divides the real line into two intervals. The solution is $x < 1$ or $x > 1$, that is any x except for $x = 1$. One can write $x \in (-\infty, 1) \cup (1, +\infty)$.

Same method can be used to solve any polynomial inequality, for example $x^n + bx^{n-1} + \dots \geq 0$, where n is greater than 2 — but we need to know the way to either find the roots of the corresponding equation or to have factorization given to us.

Example 6. Solve the inequality $(x + 1)(x - 2)^2(x - 4)^3 \geq 0$.

Solution. Notice that if we solve the corresponding equation $(x + 1)(x - 2)^2(x - 4)^3 = 0$, we get $x = -1, 2, 4$. Therefore, we need to consider the following 4 intervals: $(-\infty; -1), (-1; 2), (2; 4), (4; \infty)$.

Notice that in the 1st interval, the expression $(x + 1)(x - 2)^2(x - 4)^3$ is positive, and therefore satisfies the inequality.

Then, as x “crosses” point 1, the expression changes its sign to ‘-’, and therefore the interval $(-1; 2)$ does not satisfy the inequality.

Now, crossing point 2 again won’t change the sign of the expression, because $(x - 2)^2$ is always positive. Therefore, the interval $(2; 4)$ also doesn’t satisfy the inequality.

Finally, crossing point 4, the expression changes its sign to ‘+’, and therefore the interval $(4; \infty)$ satisfies the inequality. So, the answer to the inequality is:

$$x \in (-\infty; -1] \cup 2 \cup [4; \infty)$$

The method used to solve this problem is called a **snake method**.

Example 7. Solve the inequality $\frac{(x+1)(x-2)^2}{(x-4)^3} \geq 0$.

Solution. Note that the factors in this inequality are exactly the same as in the previous example, so the solution will be the same with the small (but important) exception: the denominator cannot be equal to 0, and therefore, x cannot be equal to 4 — notice the round instead of square bracket in the answer!

$$x \in (-\infty; -1] \cup 2 \cup (4; \infty)$$

INEQUALITIES WITH ABSOLUTE VALUE

When you have an inequality with absolute value, you will have to consider various cases: when the expression under absolute value is positive and when the expression under the absolute value is negative, and use the definition of the absolute value:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } -x \geq 0 \end{cases}$$

Example 8. Solve inequality $|x - 4| < 7$.

Solution. Solution: Again, as before, we need to consider two cases, the one when $x - 4 \geq 0$ and the one when $x - 4 < 0$.

Case 1. $x - 4 \geq 0$ means that $x \geq 4$. Now, since $x - 4 \geq 0$, we have $|x - 4| = x - 4$, and the inequality can be rewritten as

$$x - 4 < 7$$

Solving this inequality gives us $x < 11$. But remember, x must be greater than or equal to 4! So, combining both things together, we get $4 \leq x < 11$, or $x \in [4; 11)$.

Case 2. $x - 4 < 0$ means that $x < 4$. Now, since $x - 4 < 0$, we have $|x - 4| = -(x - 4) = 4 - x$, and the inequality can be rewritten as

$$4 - x < 7$$

Solving this inequality gives us $x > -3$. But remember, x must also be less than 4! So, combining both things together, we get $-3 < x \leq 4$.

Combining Cases 1 and 2 together, we get the final solution to the inequality: $-3 < x < 11$ or

$$x \in (-3, 11)$$

HOMEWORK

1. Solve the following equations.

(a) $|x - 3| = 5$

(b) $|2x - 1| = 7$

(c) $|x^2 - 5| = 4$

2. Solve the following equations.

(a) $\frac{(x+1)}{(x-1)} = 3$

(b) $\frac{(x^2 - 9)}{(x+1)} = (x+3)$

(c) $x - \frac{3}{x} = \frac{5}{x} - x$

3. Solve the following inequalities, show solution on the real line, write the answer in the interval notation.

(a) $|x - 2| > 3$

(b) $|x - 1| > x + 3$

(c) $\frac{(x-2)}{(x+3)} \leq 3$

4. Solve the following quadratic equations and inequalities:

(a) $x^2 + 2x - 3 = 0$, $x^2 + 2x - 3 > 0$, $x^2 + 2x - 3 \leq 0$

(b) $x^2 + 2x + 3 = 0$, $x^2 + 2x + 3 \geq 0$, $x^2 + 2x + 3 < 0$

(c) $-x^2 + 6x - 9 = 0$, $-x^2 + 6x - 9 \geq 0$, $-x^2 + 6x - 9 < 0$

(d) $3x^2 + x - 1 = 0$, $3x^2 + x - 1 \geq 0$, $3x^2 + x - 1 \leq 0$

5. Solve the following inequalities, using the snake method. Show solution on the real line. Write the answer in the interval notation.

(a) $(x-1)(x+2) > 0$

(b) $(x+3)(x-2)^2 \leq 0$

(c) $x(x-1)(x+2) \geq 0$

(d) $x^2(x+1)^5(x+2)^3 > 0$

*6. Solve the following inequalities, using the snake method. Show solution on the real line. Write the answer in the interval notation.

(a) $|x^2 - x| \geq 2x$

(b) $\frac{x(x-1)^2}{(x+1)^2} \geq 0$