MATH 7: HANDOUT 17
INEQUALITIES.

INEQUALITIES

Previously we were concentrating on solving equations. We can also consider **inequalities**, where instead of equal sign we have $\leq, <, \geq$, or $>$ signs. Similarly to solving equations, when we solve inequalities, we want to figure out *all* valued of x that make the inequality true.

Linear inequalities are similar to linear equations: they have parts with x and numbers only. When solving linear inequalities we use the same techniques as those used for solving linear equations. The important exception to this is that when multiplying or dividing by a negative number, you must reverse the direction of the inequality.

Example 1. $x + 2 \leq 5$.

Solution. *Subtracting* 2 *from both sides, we get* $x \leq 3$. *Therefore, any value of* x *less than or equal to* 3 *would make this inequality true. We can write this solution using interval notation:* $(-\infty, 3]$.

Notice, that **square bracket** represents that the ends of intervals that are **included**. If we don't want to include the end of the interval, we would use **round parentheses**.

Example 2. $5 - 3x > 23$.

Solution. *Subtracting* 5 *from both sides, we get* $-3x > 18$. *Now, dividing by* -3, *we need to reverse the sign:* $x < -6$. *Therefore, any value of* x *less than* -6 *would make the original inequality true. We can write this solution using interval notation:* $(-\infty, -6)$.

PRODUCT OF TWO EXPRESSIONS

If we have an inequality that involves a product of two expressions compared to 0, we should consider several cases:

- If the inequality if $f(x) \cdot g(x) > 0$, where $f(x)$ and $g(x)$ are some expressions involving x, then there are two cases: 1) $f(x) > 0$ and $g(x) > 0$ and 2) $f(x) < 0$ and $g(x) < 0$.
- If the inequality if $f(x) \cdot g(x) < 0$, where $f(x)$ and $g(x)$ are some expressions involving x, then there are two cases: 1) $f(x) < 0$ and $g(x) > 0$ and 2) $f(x) > 0$ and $g(x) < 0$.

Example 3. $(x - 1)(x + 2) \geq 0$.

Solution. *A product of two expressions is positive in two cases: 1) both expressions are positive; 2) both expressions are negative. Let us consider them separately:*

Case 1: $x - 1 \leq 0;\ x + 2 \leq 0$. *This comes down to two inequalities:* $x \leq 1;\ x \leq -2$. *For both of them to be true,* x *should be below* -2*:* $x \leq -2$, *or* $(-\infty, -2]$ *in interval notation..*

Case 2: $x - 1 \geq 0;\ x + 2 \geq 0$. *This comes down to two inequalities:* $x \geq 1;\ x \geq -2$. *For both of them to be true,* x *should be above* 1*:* $x \geq 1$, *or* $[1, \infty)$ *in interval notation.*

The final solution is $x \leq -2$ *or* $x \geq 1$. *We can use a union of intervals to write the solution:* $(-\infty, -2] \cup [1, \infty)$.

RATIONAL INEQUALITIES

A similar approach that we discussed in the previous section works for fractions of two expressions involving x.

Example 4. $\dfrac{x - 4}{2x - 10} \leq 0$.

Solution. *Notice that we can't multiply both parts by* $2x - 10$, *since we don't know whether it's positive or negative — multiplication by a negative expression would change the sign, so we should be careful. We will use a different method. A fraction is negative in two cases: 1) numerator is positive, denominator is negative; 2) numerator is negative, denominator is positive. Let us consider them separately:*

Case 1: $x - 4 \geq 0;\ 2x - 10 < 0$. *This comes down to two inequalities:* $x \geq 4;\ x < 5$. *For both of them to be true, x should be between* 4 *and* 5*:* $4 \leq x < 5$.

Case 2: $x - 4 \leq 0;\ 2x - 10 > 0$. *This comes down to two inequalities* $x \leq 4;\ x > 5$. *Both of them cannot be true simultaneously, so this case is impossible.*

The final solution is $4 \leq x < 5$*, or* $[4, 5)$.

Example 5. $\dfrac{2x-1}{x+3} \geq 3.$

Solution. *Notice that as before we can't multiply both parts by* $x + 3$*, since we don't know whether it's positive or negative — multiplication by a negative expression would change the sign, so we should be careful.*

Therefore, we will subtract 3 *from both sides:*

$$\begin{aligned} \frac{2x-1}{x+3} &\geq 3 \\ \frac{2x-1}{x+3} - 3 &\geq 0 \\ \frac{(2x-1) - 3(x+3)}{x+3} &\geq 0 \\ \frac{-x-10}{x+3} &\geq 0 \end{aligned}$$

Now, a fraction is positive in two cases: 1) numerator is positive, denominator is positive; 2) numerator is negative, denominator is negative. Let us consider them separately:

Case 1: $-x - 10 \geq 0;\ x + 3 > 0$. *This comes down to two inequalities:* $x \leq -10;\ x > -3$. *Both of them cannot be true simultaneously, so this case is impossible.*

Case 2: $-x - 10 \leq 0;\ x + 3 < 0$. *This comes down to two inequalities* $x \geq -10;\ x < -3$. *For both of them to be true,* x *should be between* -10 *and* -3*:* $-10 \leq x < -3$*, or* $[-10, -3)$.

The final solution is $-10 \leq x < -3$*, or* $[-10, -3)$.

HOMEWORK

1. Solve the following linear inequalities:

(a) $4x + 6 < 2x + 14$ (b) $-2(x + 3) < 10$ (c) $-2(x + 2) > 4 - x$

2. Solve the following rational inequalities:

(a) $\dfrac{1 + 3x}{x - 2} > 2$ (b) $\dfrac{x + 1}{x - 5} \leq 0$ (c) $\dfrac{3x + 1}{x + 4} \geq 1$

3. Find all values of a so that the following equation has a positive solution. [**Hint:** Express x in terms of a, and then find the values of a so that the resulting expression is positive]

$$4 - a = \frac{2}{x - 1}$$

4. Solve the following inequalities:

(a) $x^2 \leq 81$
(b) $x^2 \geq 100$
(c) $\sqrt{x} \leq 7$
(d) $\sqrt{x + 10} < 2$