MATH 7: HANDOUT 16
ABSOLUTE VALUES.

Equations with Absolute Value

The absolute value of a number, denoted as $|x|$, represents the distance of x from zero on the number line. It is always non-negative.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

When solving equations involving absolute values, it is important to consider both the positive and negative cases for the expression inside the absolute value.

Basic Steps for Solving Absolute Value Equations

- **Isolate the absolute value expression**: Ensure $|f(x)|$ is by itself on one side of the equation.
- **Set up two cases**:
  - Case 1: $f(x) = k$
  - Case 2: $f(x) = -k$, where $k \geq 0$
- **Solve each case**.
- **Check for extraneous solutions**: Verify that the solutions satisfy the original equation.

Critical Points and Domain

When working with absolute value equations or expressions involving multiple absolute values, the **critical points** are the values of x that make any expression inside an absolute value equal to zero. These critical points divide the number line into intervals, within which the behavior of the absolute value expressions is consistent.

Finding Critical Points. For an expression like $|f(x)| + |g(x)|$, find the values of x that satisfy:

- $f(x) = 0$
- $g(x) = 0$

The solutions to these equations are the critical points.

Checking the Domain. To determine the domain of the solution, analyze whether the solutions satisfy any additional conditions in the problem, such as non-negativity of the terms involved (for example, absolute value should never be negative. Always verify solutions against the original equation or inequality.

Examples

Example 1. Solve: $|x| = 5$

Solution:

- Case 1: $x = 5$
- Case 2: $x = -5$

Answer: $x = \pm 5$

Example 2. Solve: $|x - 3| = 7$

Solution:

- Case 1: $x - 3 = 7 \implies x = 10$
- Case 2: $x - 3 = -7 \implies x = -4$

Answer: $x = 10$ or $x = -4$

Example 3. Simplify: $|x-4|+|3x+2|$

Solution:

- Identify critical points: $x=4$ and $x=-\frac{2}{3}$.
- Case 1: $x \geq 4$: Simplify $(x-4)+(3x+2)=4x-2$.
- Case 2: $-\frac{2}{3} \leq x < 4$: Simplify $-(x-4)+(3x+2)=2x+6$.
- Case 3: $x < -\frac{2}{3}$: Simplify $-(x-4)-(3x+2)=-4x+2$.

Answer: Based on the value of x, the answer can be written in piecewise form:

$$|x-4|+|3x+2| = \begin{cases} 4x-2, & x \geq 4 \\ 2x+6, & -\frac{2}{3} \leq x < 4 \\ -4x+2, & x < -\frac{2}{3} \end{cases}$$

Example 4. Solve: $|x-1|+|2x+3|=5$

Solution:

- Break into cases based on critical points: $x=1$ and $x=-\frac{3}{2}$.
- Case 1: $x \geq 1$: Solve $(x-1)+(2x+3)=5 \implies 3x+2=5 \implies x=1$.
- Case 2: $-\frac{3}{2} \leq x < 1$: Solve $-(x-1)+(2x+3)=5 \implies x=1$. Since 1 is not ≤ 1, this solution is excluded.
- Case 3: $x < -\frac{3}{2}$: Solve $-(x-1)-(2x+3)=5 \implies x=-\frac{7}{3}$. Note that $-\frac{7}{3} < -\frac{3}{2}$, so this solution is included.

Answer: $x=1$ or $x=-\frac{7}{3}$

Example 5. Solve: $|x+2|=x-4$

Solution:

- Break into cases based on the critical point: $x=-2$
- Case 1: $x \geq -2$: Solve $x+2=x-4$ — impossible, no solutions in this case.
- Case 2: $x < -2$: Solve $-x-2=x-4 \implies 2x=2 \implies x=1$. But $x=1$ does not satisfy the case condition $x < -2$, so no solutions in this case.

Answer: No solution.

Example 6. Solve: $|x^2-4|=2x$

Solution:

- Case 1: $x^2-4 \geq 0 \implies x \geq 2$ or $x \leq -2$:
  Solve $x^2-4=2x \implies x^2-2x-4=0 \implies x=1 \pm \sqrt{5}$ (using quadratic formula).
  Only $1+\sqrt{5}$ satisfies the case condition, since the other solution is between -2 and 2: $-2 < 1-\sqrt{5} < 2$.
- Case 2: $x^2-4 < 0 \implies -2 < x < 2$:
  Solve $-x^2+4=2x \implies x^2+2x-4=0 \implies x=-1 \pm \sqrt{5}$ (using quadratic formula).
  Only $-1+\sqrt{5}$ satisfies the case condition, since the other solution is less than -2: $-1-\sqrt{5} < -2$.

Domain Check: $x \geq 0$ since $2x \geq 0$.

- From Case 1: $x=1+\sqrt{5}$ satisfies $x \geq 0$.
- From Case 2: $x=-1+\sqrt{5}$ satisfies $x \geq 0$.

Answer: $x=1+\sqrt{5}, x=-1+\sqrt{5}$

Homework

1. Evaluate:
  (a) If $x = -3$, calculate $|x| + |x + 2|$.
  (b) If $x = 2$ and $y = -4$, find $|x - y| + |y - x|$.

2. Simplify (write what this expression is equal to for each value of x; use the same notation as we used in the definition of $|x|$, similar to Example 3 above.):
  (a) $|2x + 3| + |x - 1|$.
  (b) $|x + 6| + |4x - 2|$

3. Solve the following equations:
  (a) $|x + 5| = 7$
  (b) $|3x - 2| = 4$
  (c) $|x - 1| + 2 = 6$
  (d) $|x + 3| = |x - 1|$
  (e) $|x^2 - 4x + 3| = 5$
  (f) $|x + 2| + |x - 1| = 6$
  (g) $|x - 3| + |x + 1| = 10$

***4.** Solve: $|x^3 - 4x| = 3x$.

***5.** For what values of a and b does the equation $|x - a| + |x - b| = 10$ represent a line segment on the number line? Find a and b.