MATH 7: HANDOUT 26

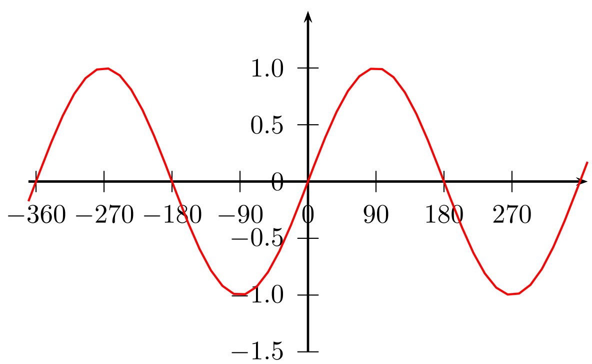
TRIGONOMETRY 4: TRIGONOMETRIC GRAPHS. TRIGONOMETRIC EQUATIONS.

Graph of sine

By looking at the values of sine as we go around the trigonometric circle, we find out a few facts like:

- $\sin 0 = \sin \pi = 0$
- $\sin x$ increases from 0 to $\frac{\pi}{2}$.
- At $x = \frac{\pi}{2}$, $\sin x$ reaches it's maximum value, 1.
- At $x = \frac{3\pi}{2}$, $\sin x$ reaches it's minimum value, -1.
- $\sin(x + 2\pi) = \sin x$.

We can see all of these facts clearly in the graph of the function $\sin x$:



Trigonometric Equations

We can use our experience solving linear and quadratic equations to solve analogous problems involving trigonometric functions.

There are important differences, however. For example, the periodicity of the trigonometric functions usually means that there are many solutions to an equation (an infinite number!). Let's look at a few examples:

Solving the equation $\sin x = \sin c$. Using the trigonometric circle (see figure 1), we see that, for a given c, the solution to the equation $\sin x = \sin c$ is

$$x = c + 360^\circ \times n$$

or

$$x = \pi - c + 360^\circ \times n,$$

where n can be any integer number (that's just because adding full turns doesn't change the sign!), which is why we have an infinite number of solutions. We can use the same technique to solve equations of the type $\sin x = a$. For example, equation $\sin x = \frac{\sqrt{3}}{2}$ has solutions

$$x = 60^\circ + 360^\circ \times n$$

or

$$x = 120^\circ + 360^\circ \times n,$$

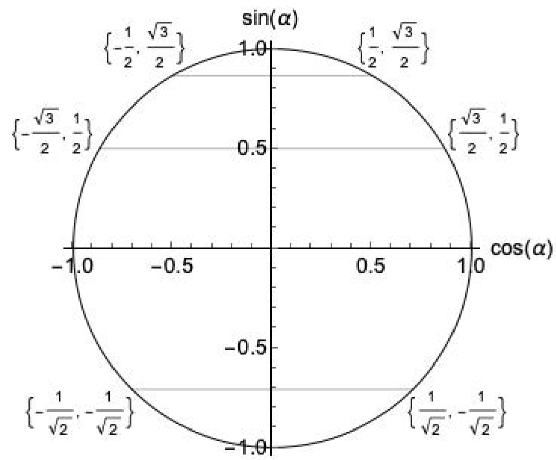


Figure 1. A few examples of the relation $\sin x = \sin\left(180^\circ - x\right)$.

Solving the equation $\cos x = \cos c$. By a similar inspection of the trigonometric circle (figure 2), we see that, for a given c, the solution to the equation $\cos x = \cos c$ is

$$x = c + 360^\circ \times n$$

or

$$x = -c + 360^\circ \times n,$$

where n can be any integer number. We can use the same technique to solve equations of the type $\cos x = a$. For example, equation $\cos x = \frac{\sqrt{2}}{2}$ has solutions

$$x = 45^\circ + 360^\circ \times n$$

or

$$x = -45^\circ + 360^\circ \times n,$$

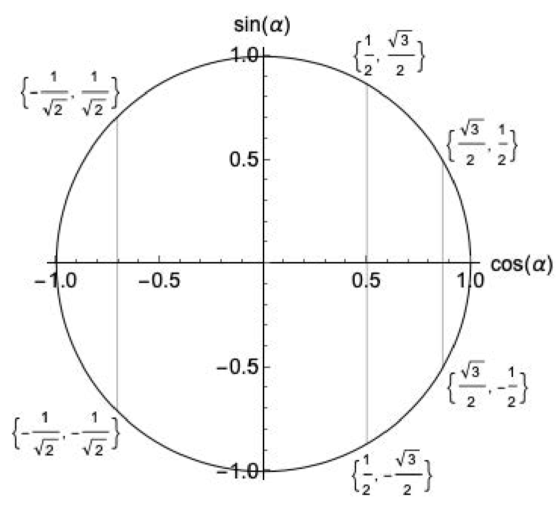


Figure 2. A few examples of the relation $\cos x = \cos\left(-x\right)$.

General Strategy. We see from these examples that, rather than trying to memorize all different cases, one should always refer to the trigonometric circle, from which the answer can be read off immediately. This is also true for trigonometric inequalities.

Tangent in the Trigonometric Circle

Just like with the sine and the cosine, the tangent can be defined through the trigonometric circle. Given a point (with coordinates $(\cos\alpha, \sin\alpha)$) corresponding to the angle α, one draws a line which passes through the origin and through this point, the tangent of α is the position along a line tangent to the circle through the point $(1,0)$ of the intersection of these two lines (see figure 3). For angles $0 < \alpha < \pi/2$, this agrees with the previous. For other angles, we will use this procedure to define the tangent.

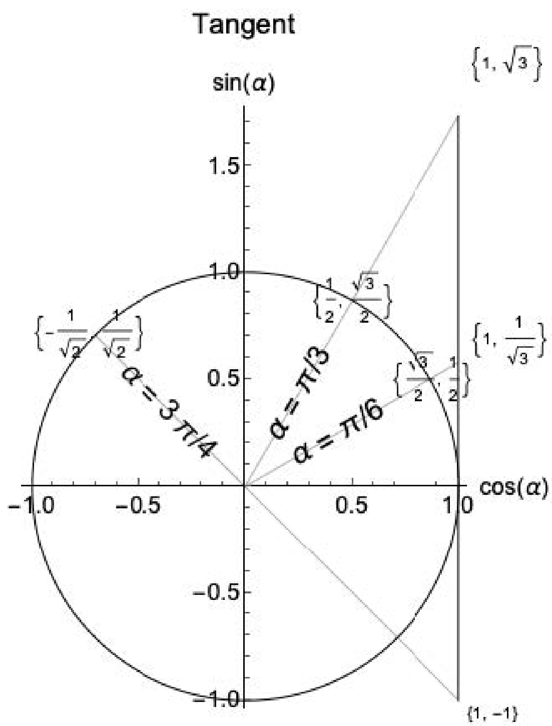


Figure 3. Meaning of $\tan\alpha$ in the trigonometric circle.

Compare this definition with the values we know from the table:

Trigonometric Functions							
Function	Notation	Definition	0	$\frac{\pi}{6}/30°$	$\frac{\pi}{4}/45°$	$\frac{\pi}{3}/60°$	$\frac{\pi}{2}/90°$
sine	$\sin(\alpha)$	$\frac{\text{opposite side}}{\text{hypotenuse}}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cosine	$\cos(\alpha)$	$\frac{\text{adjacent side}}{\text{hypotenuse}}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tangent	$\tan(\alpha)$	$\frac{\text{opposite side}}{\text{adjacent side}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

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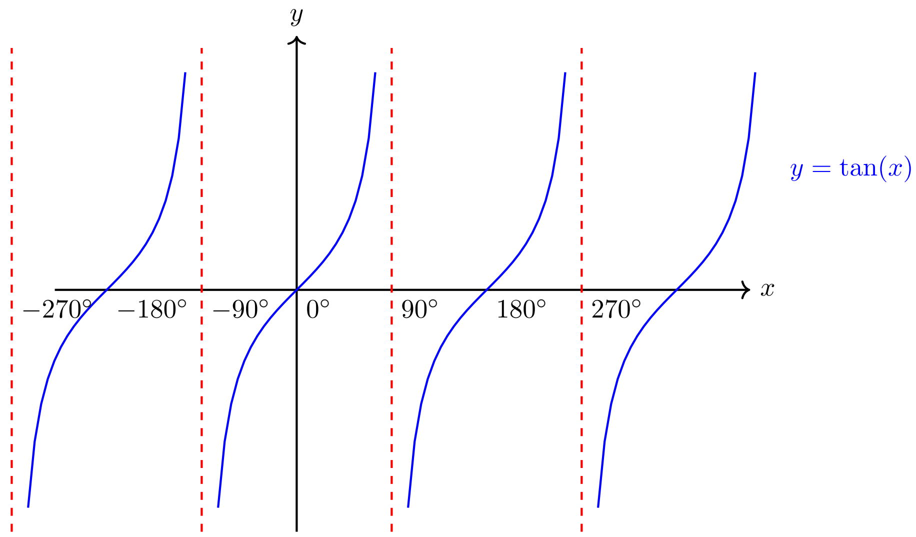
Graph of tangent

By looking at the values of tangent as we go around the trigonometric circle, we find out a few facts like:

- $\tan 0° = \tan 180° = 0$
- $\tan x$ increases from 0 to 90°.
- As x is approaching 90°, $\tan x$ grows all the way to "infinity."
- As x goes from 0° to −90°, $\tan x$ decreases all the way to "negative infinity."
- $\tan x + 180° = \tan x$.

We can see all of these facts clearly in the graph of the function $\tan x$:



Solving the equation $\tan x = \tan c$. In this case (figure 4) we find that, for a given c, the solution to the equation $\tan x = \tan c$ is

$$x = c + \pi \times n$$

or

$$x = c + 180° \times n,$$

where n can be any integer number.

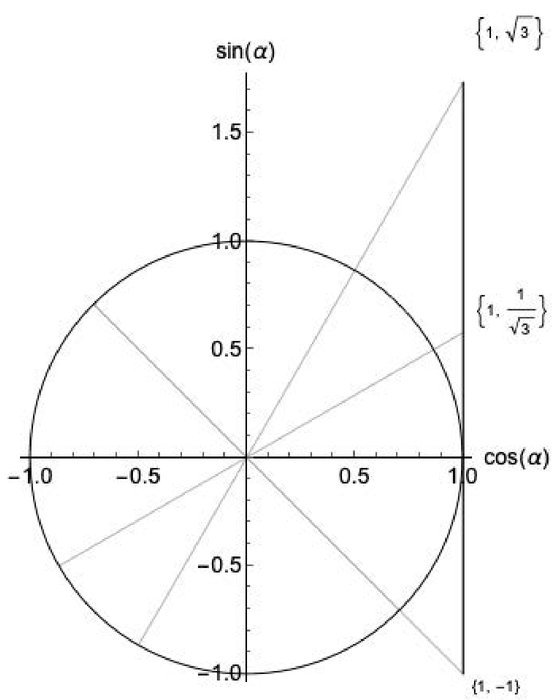


Figure 4. A few examples of the relation $\tan x = \tan (x + \pi)$.

Homework

1. Fill out the following table. Make sure you understand how to convert degrees to radians, and use the values of sine and cosine that you already know!

Degrees	Radians	sine	cosine
180°	π	0	-1
45°			
60°			
120°			
150°			
210°			
315°			
	$2\pi/3$		
	$9\pi/4$		
	$5\pi/6$		
	$-5\pi/4$		
	$11\pi/3$		
	$7\pi/6$		
		$\sqrt{3}/2$	$1/2$
		$\sqrt{2}/2$	$-\sqrt{2}/2$
		$-1/2$	$-\sqrt{3}/2$

2. Using the trigonometric circle, show that $\cos x = \sin(x + \pi/2)$ for any angle x. Then use this fact and the graph of the sine function to construct (draw) the graph of the cosine function.
3. Solve the following equations
   (a) $\sin x = \sin \frac{\pi}{5}$
   (b) $\sin x = 0$
   (c) $\cos x = \frac{\sqrt{2}}{2}$
   (d) $\cos x = \frac{1}{2}$
   (e) $\cos(x + \frac{\pi}{6}) = 0$
4. Solve the following equations
   (a) $(\sin x)^2 - \sin x = 0$ [Hint: start by defining $y = \sin x$.]
   (b) $2(\sin x)^2 - 3\sin x + 1 = 0$
   (c) $4\cos x + \frac{3}{\cos x} = 8$
5. Solve the following equations
   (a) $\sqrt{3}\tan x = 1$
   (b) $\tan x = -1$
   (c) $\tan x = -\sqrt{3}$
   (d) $(\sin x)^2 = (\cos x)^2$ [Hint: $\tan x = \sin x / \cos x$.]
6. What are the values of x for which:
   (a) $\sin x > -\frac{\sqrt{2}}{2}$
   (b) $-\frac{1}{2} \leq \sin x < \frac{\sqrt{2}}{2}$
   (c) $2(\sin x)^2 < \sin x$
   (d) $\cos 2x \leq \frac{\sqrt{3}}{2}$
   (e) $(\tan x^2)^2 \leq 1$

   It might help to examine the graphs of trigonometric functions to answer this question.