Math 7: Homework 20
Trigonometric inequalities

In the previous class, we saw how the trigonometric circle helps to solve trigonometric equations (refer to the notes on trigonometric equations): both sine and cosine values can be read off the circle or converted back into angles. All solutions to simple equations like $\sin\alpha = c$ are evident. The circle is even more handy when solving trigonometric inequalities. Like with equations, the periodicity of the trigonometric functions (arbitrary number of full rotations around the circle), there will be infinitely many intervals of the angle α satisfying the inequality, all different by an integer number of 2π (or $360°$).

SOLVING THE INEQUALITY $\sin\alpha > \sin c$

To solve this inequality, we need to find *all points of the circle such that* $y > \sin c$. In the example in Figure 1, $\sin c = \frac{1}{2}$. All the points on the circle with $y > \frac{1}{2}$ will satisfy the inequality; it is clear that they correspond to angles α

$$\frac{\pi}{6} < \alpha < \frac{5\pi}{6}$$

but also, all the angles that are different from these by any multiple of (2π). Just as before, we will summarize all such intervals using an arbitrary integer number $(n \in \mathbb{Z})$:

$$2\pi n + \frac{\pi}{6} < \alpha < 2\pi n + \frac{5\pi}{6}\,, \quad n \in \mathbb{Z} \text{ (that is, any integer } n\text{)}\,.$$

This sequence of inequalities simply says that the complete solution is the *union of an infinite number of intervals*, thanks to $\sin\alpha$ being periodic:

$$\alpha \in \ldots \cup (-2\pi + \frac{\pi}{6}; -2\pi + \frac{5\pi}{6}) \cup (\frac{\pi}{6}; \frac{5\pi}{6}) \cup (2\pi + \frac{\pi}{6}; 2\pi + \frac{5\pi}{6}) \cup (4\pi + \frac{\pi}{6}; 4\pi + \frac{5\pi}{6}) \cup \ldots\,.$$

The $(\ldots)$ symbol above on the left stands for all the intervals for negative α's and $(\ldots)$ to the right stands for the same for positive α's.

Note that the opposite inequality $\sin\alpha < \frac{1}{2}$ (the right figure) spans the other part of the circle, $\frac{5\pi}{6} < \alpha < \frac{13\pi}{6}$, and with an arbitrary number of full rotations (2π),

$$2\pi n + \frac{5\pi}{6} < \alpha < 2\pi n + \frac{13\pi}{6}\,, \quad n \in \mathbb{Z}$$

Let's write this answer in an equivalent form using negative angle $(-7\pi/6)$ instead of $5\pi/6$:

$$2\pi n - \frac{7\pi}{6} < \alpha < 2\pi n + \frac{\pi}{6}\,, \quad n \in \mathbb{Z}$$

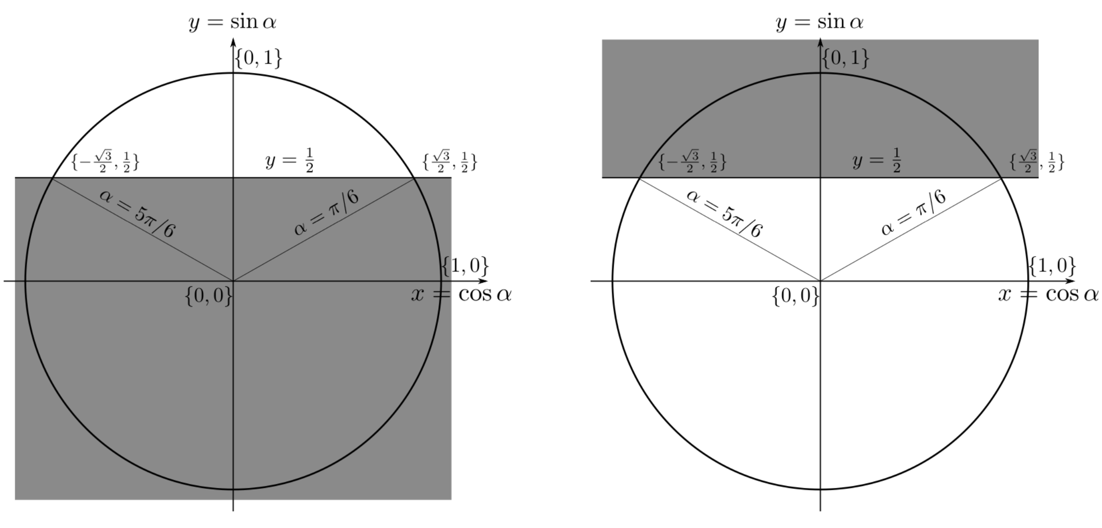

FIGURE 1. Solving equations $\sin\alpha > \frac{1}{2}$ (left) and $\sin\alpha < \frac{1}{2}$ (right): points on the circle in the unshaded regions satisfy the inequalities.

By analogy, for the inequalities with arbitrary right-hand side $\sin c$, the solutions are (assuming that $-\frac{\pi}{2} < c < \frac{\pi}{2}$!)

$$\begin{aligned} \sin\alpha > \sin c &\iff 2\pi n + c < \alpha < 2\pi n + \pi - c\,, \quad n \in \mathbb{Z}\,, \\ \sin\alpha < \sin c &\iff 2\pi n - \pi - c < \alpha < 2\pi n + c\,, \quad n \in \mathbb{Z}\,. \end{aligned}$$

SOLVING THE INEQUALITY $\cos\alpha > \cos c$

By a similar inspection of the trigonometric circle (Figure 2), we see that, for example, the solution to the inequality $\cos\alpha > \frac{\sqrt{2}}{2}$ is

$$\cos\alpha > \frac{\sqrt{2}}{2} \iff 2\pi n - \frac{\pi}{4} < \alpha < 2\pi n + \frac{\pi}{4}\,, \quad n \in \mathbb{Z}$$

and the solution to the opposite inequality

$$\cos\alpha < \frac{\sqrt{2}}{2} \iff 2\pi n + \frac{\pi}{4} < \alpha < 2\pi n + \frac{7\pi}{4}\,, \quad n \in \mathbb{Z}$$

In general, if the inequality is $\cos\alpha > \cos c$ or $\cos\alpha < \cos c$, then (assuming $0 < c < \pi$!)

$$\begin{aligned} \cos\alpha > \cos c &\iff 2\pi n - c < \alpha < 2\pi n + c\,, \quad n \in \mathbb{Z} \\ \cos\alpha < \cos c &\iff 2\pi n + c < \alpha < 2\pi n + 2\pi - c\,, \quad n \in \mathbb{Z}\,. \end{aligned}$$

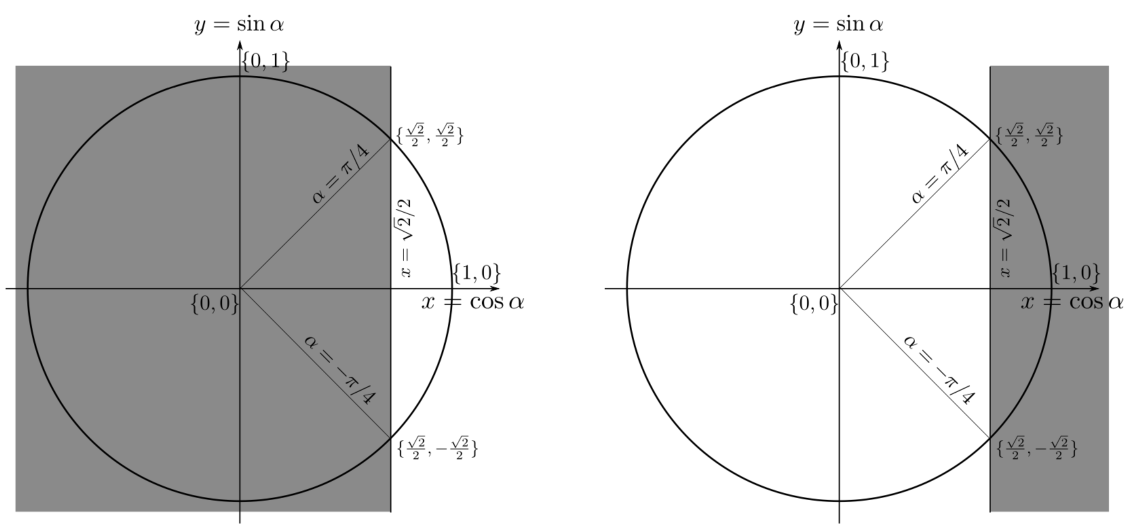

FIGURE 2. Solving equations $\cos\alpha > \frac{\sqrt{2}}{2}$ (left) and $\cos\alpha < \frac{\sqrt{2}}{2}$ (right): points on the circle in the unshaded regions satisfy the inequalities.

HOMEWORK

1. Solve the following equations [We solved these problems in class]
   (a) $(\sin x)^2 - \sin x = 0$ [Hint: start by defining $y = \sin x$.]
   (b) $2(\sin x)^2 - 3\sin x + 1 = 0$
   (c) $4\cos x + \frac{3}{\cos x} = 8$

2. Solve the following inequalities :
   (a) $\sin\alpha > \frac{\sqrt{3}}{2}$;
   (b) $\sin\alpha < -\frac{1}{2}$;
   (c) $\cos(2\alpha) > 0$;
   (d) $\cos\alpha < -\frac{\sqrt{2}}{2}$.

3. Solve the following inequalities and systems of inequalities:
   (a) $(\sin\alpha)^2 - \frac{1}{2}\sin\alpha < 0$ [start by defining $y = \sin\alpha$] ;
   (b) $|\sin(2\alpha)| > \frac{\sqrt{3}}{2}$;

(c) $\begin{cases} \cos\alpha < -\frac{1}{2} \\ \sin\alpha > \frac{1}{2} \end{cases}$.