MATH 7: HOMEWORK 12
General quadratic formula
January 12, 2025

1. Quadratic equation in a standard form.

Standard form: $ax^2 + bx + c = 0$

A **quadratic equation** could have

- no solution,
- one solution,
- two solutions depending on the coefficients a, b, and c.

Factored form: $(x - x_1)(x - x_2) = 0$, where x_1 and x_2 are the solutions of the equation, also known as *roots.*

2. Solving the incomplete quadratic equation by factorizing.

- When $c = 0$, $ax^2 + bx = 0$

  $x(ax + b) = 0$ The two roots are $x_1 = 0$ and $x_2 = -b/a$
- When $b = 0$, $ax^2 + c = 0$

  If c < 0, factorize the equation using the formula for fast multiplication $a^2 - b^2 = (a - b)(a + b)$. (*)

For example, $x^2 - 25 = 0 \Rightarrow x^2 - 5^2 = 0 \Rightarrow (x - 5)(x + 5) = 0$. $x = \pm 5$

If c > 0, there are no real solutions. An easy way to see this is to solve directly for x: $x^2 + 25 = 0 \Rightarrow x^2 = -5^2$; No number squared is equal to a negative number!

2. Solving the complete quadratic equation

- By completing the square

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 9 + 2 = (x + 3)^2 - 7 = (x + 3)^2 - \left(\sqrt{7}\right)^2 = \left(x + 3 + \sqrt{7}\right)\left(x + 3 - \sqrt{7}\right)$$

Thus, $x^2+6x+2 = 0$ if and only if $(x+3+\sqrt{7}) = 0$, which gives $x = -3-\sqrt{7}$, or $(x+3-\sqrt{7}) = 0$, which gives $x = -3+\sqrt{7}$.

- By using the quadratic formula

Completing the square works in general for any quadratic equation in a standard form

If $a = 1$, then:

$$x^2 + bx + c = x^2 + 2\frac{b}{2}x + c = \left(x^2 + 2\frac{b}{2}x + \frac{b^2}{2^2}\right) - \frac{b^2}{2^2} + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{2^2} = \left(x + \frac{b}{2}\right)^2 - \frac{D}{2^2} \qquad \text{eq (1)}$$

Thus $x^2 + bx + c = 0$ is equivalent to: $\left(x + \frac{b}{2}\right)^2 = \frac{D}{4}$

If $a \neq 1$, then: $ax^2 + bx + c = 0$ divide by a $\Longrightarrow$ $\boldsymbol{x^2 + \frac{b}{a}x + \frac{c}{a} = 0}$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{2^2a^2}\right) - \frac{b^2}{2^2a^2} + c = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{2^2a^2}$$

is equivalent to: $\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}$, where $\boldsymbol{D = b^2 - 4ac}$

The determinant D determines the number of solutions. D < 0, there are no real solutions; if D = 0, there is one solution,

if $D > 0$, the solutions are: $x + \frac{b}{2a} = \pm\sqrt{\frac{D}{4a^2}}$

$$\boldsymbol{x = \frac{-b \pm \sqrt{D}}{2a}} \qquad \boldsymbol{D = b^2 - 4ac} \qquad \text{eq (2)}$$

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

Note: Use the formulas for fast multiplication $a^2 - b^2 = (a-b)(a+b)$, $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

1. This problem requires that you carefully check your work and think:
   a. Use formula (1) to prove that for any x, $x^2 + bx + c \geq -D/4$, with equality only when $x = -b/2$.
   b. Find the minimal possible value of the expression $x^2 + 4x + 2$ [Hint: use part a) or complete the square]
   c. Given a number $a > 0$, find the maximal possible value of the expression $x(a - x)$ (the answer will depend on the value or values of a. In this case, a is called a *parameter*).

2. Complete the square and solve the quadratic equations: using $(a \pm b)^2 = a^2 \pm 2ab + b^2$ $\text{and then } a^2 - b^2 = (a-b)(a+b)$
   a. $x^2 - 2x - 3 = 0$
   b. $x^2 + 8x - 9 = 0$

3. Solve the following equations. Carefully think what method you will use and write all steps in your solution. The following questions may help you: is the equation in a standard or in a factored form?; what are the coefficients a, b, c? Are some of these coefficients zero? Shall I factorize or use the quadratic formula from eq (2)?
   a. $x^2 - 5x + 5 = 0$
   b. $x^2 = 1 + x$
   c. $-4x^2 + 8x + 21 = 0$
   d. $2x(3 - x) = 1$
   e. $x^3 + 4x^2 - 45x = 0$
   f. $\frac{x}{x-2} = x - 1$

4. Indian mathematicians were aware of the quadratic formula for solving quadratic equations. Can you solve the following problem by the 9th century mathematician Mahavıra? (translated from original text)
   One-third of a herd of elephants and three times the square root of the remaining part of the herd were seen on a mountain slope; and in a lake was seen a male elephant along with three female elephants constituting the ultimate remainder. How many were the elephants here?

5. Use eq (2) to solve these equations:
   a. $4x^2 - 58 + 5 = 0$
   b. $2x^2 + 5x + 3 = 0$

6. Determine the number of solutions of the following equations. You do not need to solve them!
   a. $2x^2 + 5x - 1 = 0$
   b. $3x^2 - 4x + 10 = 0$
   c. $3x^2 - 24x + 48 = 0$
   d. $5x^2 + 7x + 6 = 0$

7. (*) Solve the following equations using Vieta's formulas:
   a. $x^3 - 2x^2 - 5x + 6 = 0$
   b. $x^3 + 6x^2 + 5x - 12 = 0$