Math 6: Homework 2.13
Invariants and semi-invariants

Invariant: something that remains unchanged.
Semi-invariant: something that changes in one direction only (for example, a number that always increases).

1) Numbers 1 through 20 are written on the blackboard. Every minute two of the numbers are erased and replaced by their sum. Can you predict which number will be written on the board at the end?

2) 100 different numbers are written in a row on the board. Every second you can do the following operation: if two numbers next to each other are in a wrong order (i.e., the larger number is before the smaller number), you can exchange them. Show that after some time all numbers will be in the correct order.

3) a) We are given a 4 × 4 table, each cell containing either + sign or − sign:

+ − + +
+ + + +
+ + + +
+ − − +

You can reverse all signs in a single row or column, replacing each + by − and − by +. Is it possible to make all signs + by repeating this operation?

(b) Same question, but for this table:

+ − + +
+ + + +
+ + + +
+ − + +

4) Each of n countries is ruled by one of two parties (let us call them "red" and "blue"). If a country A is ruled by a red party, but majority (more than a half) of its neighbors are ruled by blue party, then the blue party can use their support to overthrow the government of the country A, so that A also becomes ruled by blue party. Of course, the same applies to the red party. Show that it cannot continue indefinitely: after some time, no more regime changes will be possible.

5) There are 16 glasses on a table, arranged in a 4x4 grid, the glass in the bottom-left corner upside down. You are allowed to turn over any 1x4 row or column of glasses at a time. Can you get all the glasses standing correctly except the one in the top-right corner?