Math battle
Dec 15, 2024

Main Algebraic Identities/formula

$$a^{-n} = \frac{1}{a^n}$$

$$(a^m)^n = a^{mn}$$

$$\frac{m}{n} = \sqrt[n]{a^m}$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$a^2 - b^2 = (a-b)(a+b)$$

Arithmetic series

$$a_n = a_1 + (n-1)d$$

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

$$d = \frac{a_s - a_t}{s - t}$$

$$S = \frac{(a_1 + a_n) \times n}{2}$$

Geometric series

$$a_n = a_1 \times q^{n-1}$$

$$a_n = \sqrt{a_{n-1}.a_{n+1}}$$

$$S_n = a_1 \times \frac{(1 - q^n)}{1 - q}$$

$$\mathrm{S} = \frac{a_1}{1 - q}$$

Binomial coefficients

$nC_k = \binom{n}{k}$ = the number of paths on the chessboard going k units up and n − k to the right
= the number of words that can be written using k ones and n − k zeroes
= the number of ways to choose k items out of n (***order doesn't matter***)

- Formula for binomial coefficients

There is an explicit formula to calculate $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!}$$

- Formula for permutations (the number of ways of choosing k items out of n when ***the order matters***): Compare it with the number of ways of choosing k items out of n when the order matters:

$$nPk = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

Problems:

1. Expand as sums of powers of x : $(2x^2-3x+2)^2$
2. If $x + \frac{1}{x} = 7$, find $x^2 + \frac{1}{x^2}$ [Hint: try completing the square]
3. Factorize: (i.e., write as a product) the following expressions:
   a. $a^4 - 169b^4$ (b) $-x^4 + x^2 + 12$ [Hint: Imagine $x^2 = y$]
4. Simplify:
   a. $\frac{x}{(x^2-y^2)} - \frac{y}{(x+y)^2}$
   b. $\frac{a+b}{(b-c)(c-a)} + \frac{b+c}{(c-a)(a-b)} + \frac{c+a}{(a-b)(b-c)}$
5. The 3-rd term of the arithmetic progression is equal to 1. The 10-th term of it is three times as much as the 6-th term. Find the first term and the common difference. (**Hint:** Use the formula for the n-th term of the progression and write what is given in the problem using this formula.)
6. Calculate the sum of series: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + .. \frac{1}{2^{10}}$. What is the sum if the series is infinite?
7. How many ways are there to seat 5 students in a class that has 5 desks? if there are 10 desks?
8. The guidelines at a certain college specify that for the introductory English class, the professor may choose one of 3 specified novels and choose two from a list of 5 specified plays. Thus, the reading list for this introductory class must have one novel and two plays. How many different reading lists could a professor create within these parameters?
9. Knights and Knave problem:

   There are two native islanders, named Alice and Bob, standing next to each other. You do not know what type either of them is. Suddenly, Alice says “At least one of us is a Knave.” What are Alice and Bob?

10. You meet two inhabitants: Sally and Zippy. Sally claims, ‘I and Zippy are not the same.’ Zippy says, ‘Of I and Sally, exactly one is a knight.’ Can you determine who is a knight and who is a knave?