Homework 5: Geometric sequences

HW5 is Due November 3.

1. Geometric sequence (progression)

A sequence of numbers is a geometric progression if the next number in the sequence is the current number times a constant called the **common ratio**, let's call it **q**.

For example: 6, 12, 24, 48, ... The common ratio here is q = 2.

Sequence elements (terms) are labeled according to their position in the sequence using a counter **n** as a subscript. The value of the n-th element in a sequence is labeled as $\mathbf{a_n}$. Then, the first term in the sequence has n = 1 and a value of $a_1 = 6$, the second element is $a_2 = 15$, and so on.

We could find any element of a sequence knowing the first element a_1 and the ration q.

Example: *What is b_{10}? What is the n^{th} term?*

$a_1 = 6$

$a_2 = a_1 \times q = 6 \times 2 = 12$

$a_3 = a_2 \times q = (a_1 \times q) \times q = a_1 \times q^2 = 6 \times 2^2 = 24$

$a_4 = a_3 \times q = (a_1 \times q^2) \times q^2 = a_1 \times q^3 = 6 \times 2^3 = 48$

....

$\mathbf{a_n = a_1 \times q^{n-1}}$

So $a_{10} = a_1 \times q^9 = 6 \times 2^9 = 6 \times 512 = 3072$

2.Property of a geometric sequence

A property of a geometric sequence is that any term is geometric mean of its neighbors or any two equally distanced neighbors.

$$a_n = \sqrt{a_{n-1}.a_{n+1}} \quad = \quad \sqrt{a_{n-\mathrm{k}}.a_{n+k}}$$

Proof:

$$a_n = a_{n-1} \times q$$

$$a_n = a_{n+1} \div q$$

Multiplying these two equalities gives us:

$$a_n^2 = a_{n-1} \cdot a_{n+1}$$

from where we can get what we need.

3. **Sum of a geometric sequence,**

a) Sum of the first n-terms:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = a_1 \times \frac{(1-q^n)}{1-q}$$

Proof: To prove this, we write the sum and we multiply it by q:

$$S = a_1 + a_2 + a_3 + \cdots \mathrm{a}_{\mathrm{n}-1} + a_n$$
$$qS = qa_1 + \mathrm{q}a_2 + \mathrm{q}a_3 + \cdots + qa_{n-1} + qa_n$$

Remember that $qa_{n-1} = a_n$, so that the last term is $qa_n = q \times (a_1 \times q^{n-1}) = a_1 \times q^n$:

$$S = a_1 + a_2 + a_3 + \cdots \mathrm{a}_{\mathrm{n}-1} + a_n$$
$$qS = a_2 + a_3 + a_4 + \cdots + a_n + a_{n+1}$$

Subtracting the second equality from the first, and canceling out the terms, we get:

$$S_n - qS_n = (a_1 - a_{n+1}), \quad \text{or}$$

$$S_n(1-q) = (a_1 - a_1 q^n)$$

$$S_n(1-q) = a_1(1-q^n)$$

from which we get the formula above.

b) Sum of **Infinite Sum**

If $0 < q < 1$, then the sum of the geometric progression is approaching some numbers, which we can call a **sum of an infinite geometric progression**, or just an **infinite sum**.

For example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ \ldots \ = \ 2$$

The formula for the infinite sum is the following: $\mathrm{S} \ = \ \frac{a_1}{1-q}$

4. Geometric sequences -summary

$$a_n = a_1 \times q^{n-1}$$

$$a_n = \sqrt{a_{n-1} . a_{n+1}}$$

$$S_n = a_1 \times \frac{(1-q^n)}{1-q}$$

$$\mathrm{S} \ = \ \frac{a_1}{1-q}$$

Homework problems are on the next page:

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

1. Write the first 5 terms of a geometric progression if $a_1 = -20$ and $q = \frac{1}{2}$
2. What are the first 2 terms of the geometric progression: a_1, a_2, 24, 36, 54, ...?
3. What is the common ratio of the geometric progression: $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots$? What is a_{10}? What is a_{100}?
4. Calculate the sum: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{10}}$
5. What is the sum of : $1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 + \cdots - 2^{15}$?
6. What is the sum of: $1 + x + x^2 + x^3 + x^4 + x^5 + \cdots + x^{100}$?
7. A geometric progression has 99 terms, the first term is 12 and the last term is 48. What is the 50-th term?