MATH 6
HANDOUT 4: MORE ABOUT IF

Basic logic operations

For you convenience, here is the list of logic operations we had used:

- NOT A: true if A is false, false if A is true
- A AND B: true if both A and B are true, false otherwise
- A OR B: true if at least one of A and B is true, false otherwise
- A XOR B: true if exactly one of A and B is true, false otherwise

If

Recall that we are studying logic rules, in particular logic rules involving operation $\Rightarrow$ (reads "implies", or "if A then B"). The truth table for $\Rightarrow$ is given below:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

That is, you can derive anything from the FALSE statement, but only truthful statements can be derived from TRUE statements.

As a reasonable example to understand it, think about the following statement: *If it rains, Mrs. Smith always carries her umbrella*, which can be rewritten as (Rain $\Rightarrow$ Umbrella).

Now, if it's raining (Rain = TRUE), and you meet Mrs. Smith with an umbrella (Umbrella = TRUE), the statement above is obviously TRUE — look at the first line of the truth table above.

If it is not raining (Rain = FALSE), and you meet Mrs. Smith with or without an umbrella (Umbrella = TRUE or FALSE), you still cannot prove the statement above to be a lie: in fact, the statement tells us nothing about the behavior of Mrs. Smith when it's not raining; as a result, the statement is TRUE again — that corresponds to the two last lines of the truth table above.

The only time you would prove the statement above to be a falsehood, is if you meet Mrs. Smith during rainy weather and without an umbrella, that is, when Rain = TRUE, and Umbrella = FALSE – the second line of the table above, the only one that results in FALSE.

Here are some of the more important rules about $\Rightarrow$:

- $A \Rightarrow B$ and $B \Rightarrow A$ are not equivalent: it is possible that one statement is true and the other is false.
- Contrapositive rule: $A \Rightarrow B$ is equivalent to $(\text{NOT } B) \Rightarrow (\text{NOT } A)$.

This construction is very useful in deducing new results from known ones. Here are some of the rules:

- Given $A \Rightarrow B$ and $B \Rightarrow C$, we can conclude $A \Rightarrow C$
- Given $A \Rightarrow B$ and NOT B, we can conclude NOT A
- Given $A \Rightarrow$ FALSE, we can conclude NOT A (proof by contradiction — if we assume something, and reach a FALSE conclusion, out assumption was wrong.)

HOMEWORK

1. If today is Thursday, then Jane's class has library day. If Jane's class has library day, then Jane will bring home new library books. Jane brought no new library books. Therefore,...

2. If it is Tuesday and Bill is in a good mood, he goes to his favorite pub, and when he goes to his favorite pub, he comes home very late. Today Bill came home early. Therefore, ...

3. Here is another one of Lewis Carrol's puzzles.
(a) All hummingbirds are richly colored.
(b) No large birds live on honey.
(c) Birds that do not live on honey are dull in color.
Therefore, ...

4. And another one:
(a) My saucepans are the only things I have that are made of tin.
(b) I find all your presents very useful.
(c) None of my saucepans are of the slightest use.
Therefore, ...

5. Let us consider a new logical operation, called NAND, which is defined by the following truth table:

A	B	A NAND B
T	T	F
T	F	T
F	T	T
F	F	T

(a) Show that A NAND B is equivalent to NOT(A AND B) (this explains the name: NAND is short for "not and").
(b) Show that A NAND A is equivalent to NOT A.
(c) Write the truth table for (A NAND B) NAND(A NAND B).
(d) Write the truth table for (A NAND A) NAND(B NAND B).
(e) Show that any logical formula which can be written using AND, OR, NOT can also be written using only NAND.

6. On the island of knights and knaves, you meet two inhabitants, X and Y. X says, "Y is a knave". Y says, "X is a knave". Who is a knave and who is a knight?