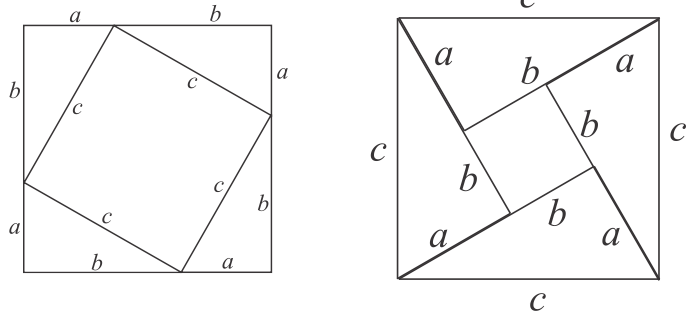


Pythagorean theorem.

4 identical right triangles are arranged as shown on the picture. The area of the big square is $S = (a + b) \cdot (a + b) = (a + b)^2$, the area of the small square is $s = c^2$. The area of 4 triangles is $4 \cdot \frac{1}{2}ab = 2ab$. But also can be represented as $S - s = 2ab$
 $2ab = (a + b) \cdot (a + b) - c^2 = a^2 + 2ab + b^2 - c^2$
 $\Rightarrow a^2 + b^2 = c^2$



Irrational numbers

Rational number is a number which can be represented as a ratio of two integers:

$$a = \frac{p}{q}; \quad p \in Z, \text{ and } q \in N, \quad (Z = \{\pm \dots, \pm 1, 0\}, N = \{1, 2, \dots\})$$

Rational numbers can be represented as infinite periodical decimals (in the case of denominators containing only powers of 2 and 5 the periodical bloc of such decimal is 0).

Numbers, which can't be expressed as a ratio (fraction) $\frac{p}{q}$ for any integers p and q are irrational numbers. Their decimal expansion is not finite, and not periodical.

Examples:

0.01001000100001000001...

0.123456789101112131415161718192021...

What side the square with the area of $a \text{ m}^2$ does have? To solve this problem, we have to find the number, which gives us a as its square. In other words, we have to solve the equation

$$x^2 = a$$

This equation can be solved (has a real number solution) only if a is nonnegative ($(a \geq 0)$) number. It can be seen very easily;

If $x = 0$, $x \cdot x = x^2 = a = 0$,

If $x > 0$, $x \cdot x = x^2 = a > 0$,

If $x < 0$, $x \cdot x = x^2 = a > 0$,

We can see that the square of any real number is a nonnegative number, or there is no such real number that has negative square.

Square root of a (real nonnegative) number a is a number, square of which is equal to a .

There are only 2 square roots from any positive number, they are equal by absolute value, but have opposite signs. The square root from 0 is 0, there is no any real square root from negative real number.

Examples:

1. Find square roots of 16: 4 and (-4) , $4^2 = (-4)^2 = 16$
2. Numbers $\frac{1}{7}$ and $(-\frac{1}{7})$ are square roots of $\frac{1}{49}$, because $\frac{1}{7} \cdot \frac{1}{7} = (-\frac{1}{7}) \cdot (-\frac{1}{7}) = \frac{1}{49}$
3. Numbers $\frac{5}{3}$ and $(-\frac{5}{3})$ are square roots of $\frac{25}{9}$, because $(\frac{5}{3})^2 = \frac{5}{3} \cdot \frac{5}{3} = (-\frac{5}{3})^2 = (-\frac{5}{3}) \cdot (-\frac{5}{3}) = \frac{25}{9}$

Arithmetic (principal) square root of a (real nonnegative) number a is a nonnegative number, square of which is equal to a .

There is a special sign for the arithmetic square root of a number a : \sqrt{a} .

Examples;

1. $\sqrt{25} = 5$, it means that arithmetic square root of 25 is 5, as a nonnegative number, square of which is 25. Square roots of 25 are 5 and (-5) , or $\pm\sqrt{25} = \pm 5$
2. Square roots of 121 are 11 and (-11) , or $\pm\sqrt{121} = \pm 11$
3. Square roots of 2 are $\pm\sqrt{2}$.
4. A few more:

$$\begin{aligned} \sqrt{0} &= 0; & \sqrt{1} &= 1; & \sqrt{4} &= 2; & \sqrt{9} &= 3; & \sqrt{16} &= 4; \\ \sqrt{25} &= 5; & \sqrt{\frac{1}{64}} &= \frac{1}{8}; & \sqrt{\frac{36}{25}} &= \frac{6}{5} \end{aligned}$$

Base on the definition of arithmetic square root we can right

$$(\sqrt{a})^2 = a$$

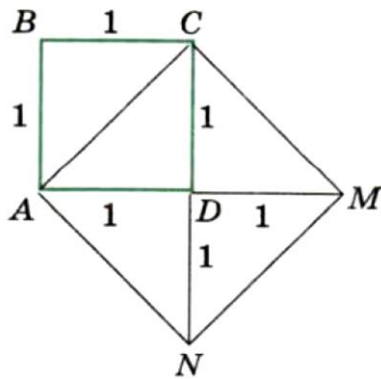
To keep our system of exponent properties consistent let's try to substitute $\sqrt{a} = a^k$. Therefore,

$$(\sqrt{a})^2 = (a^k)^2 = a^1$$

But we know that

$$(a^k)^2 = a^{2k} = a^1 \Rightarrow 2k = 1, k = \frac{1}{2}$$

To solve equation $x^2 = 23$ we have to find two sq. root of 23. $x = \pm\sqrt{23}$. 23 is not a perfect square as 4, 9, 16, 25, 36 ...



The length of the segment [AC] is $\sqrt{2}$ (from Pythagorean theorem). The area of the square ACMN is twice the area of the square ABCD. Let assume that the $\sqrt{2}$ is a rational number, so it can be represented as a ratio $\frac{p}{q}$, where $\frac{p}{q}$ is nonreducible fraction.

$$\left(\frac{p}{q}\right)^2 = 2 = \frac{p^2}{q^2}$$

Or $p^2 = 2q^2$, therefore p^2 is an even number, and p itself is an even number, and can be represented as $p = 2p_1$, consequently

$$p^2 = (2p_1)^2 = 4p_1^2 = 2q^2$$

$2p_1^2 = q^2 \Rightarrow q$ also is an even number and can be written as $q = 2q_1$.

$$\frac{p}{q} = \frac{2p_1}{2q_1}$$

therefore fraction $\frac{p}{q}$ can be reduced, which is contradict the assumption. We proved that the $\sqrt{2}$ isn't a rational number by contradiction.

$\sqrt{2}$ is an irrational number, therefore its decimal representation is an infinite nonperiodically decimal:

$$\sqrt{2} = a_0.a_1a_2 \dots$$

$$1 < 2 < 4; \Rightarrow \sqrt{1} < \sqrt{2} < \sqrt{4}; \quad 1 < \sqrt{2} < 2; \Rightarrow a_0 = 1$$

To find a_1 let's consider numbers 1.0, 1.1, 1.2, 1.3 ...

$$\begin{array}{lll} 1.0^2 = 1; & 1.1^2 = 1.21; & 1.2^2 = 1.44 \\ 1.3^2 = 1.69; & 1.4^2 = 1.96; & 1.5^2 = 2.25 \end{array}$$

Therefore:

$$1.4^2 < 2 < 1.5^2, \quad 1.4 < \sqrt{2} < 1.5$$

$$\sqrt{2} = 1.4a_2a_3 \dots$$

To find the next digit,

$$1.40^2 = 1.96; \quad 1.41^2 = 1.9881; \quad 1.42^2 = 1.2.0164$$

$$\sqrt{2} = 1.41a_3 \dots; \quad \sqrt{2} = 1.4142135 \dots$$

Exercises.

1. Simplify the following expressions (rewrite the expressions without parenthesis, combine like terms);

Example:

$$(2x + 3) \cdot (x + 7) = 2xx + 2x \cdot 7 + 3x + 3 \cdot 7 = 2x^2 + 10x + 21$$

$$(x + 5)(x + y + 3);$$

$$(k - 1 + d)(k - d);$$

$$\frac{2}{3} + 2x \left(\frac{1}{2} - \frac{1}{3}y \right) - x - \frac{1}{3}(2 - 2xy);$$

$$2x^2(x + y) - 3x^2(x - y);$$

2. Evaluate:

$$\frac{10^2 + 11^2 + 12^2 + 13^2 + 14^2}{365}$$

3. Prove that the value of the following expressions is a rational number.

a. $(\sqrt{2} - 1)(\sqrt{2} + 1)$

b. $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$

c. $(\sqrt{2} + 1)^2 + (\sqrt{2} - 1)^2$

d. $(\sqrt{7} - 1)^2 + (\sqrt{7} + 1)^2$

e. $(\sqrt{7} - 2)^2 + 4\sqrt{7}$

4. Without using calculator compare:

$$3 \dots \sqrt{11}$$

$$11 \dots \sqrt{110}$$

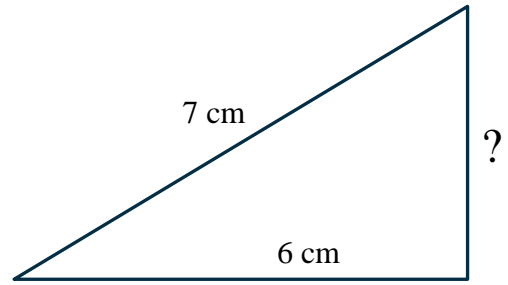
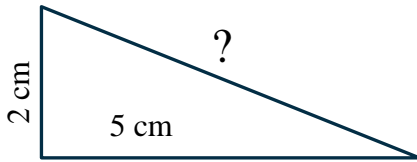
$$22 \dots \sqrt{484}$$

$$5 \dots \sqrt{20}$$

$$17 \dots \sqrt{299}$$

$$35 \dots \sqrt{1215}$$

5. Find the missing length of the side of right triangles below:



6. Evaluate:

a. $5 \cdot \sqrt{4} \cdot 3$;

c. $\sqrt{13 - 3 \cdot 3}$;

e. $\frac{1}{2} \sqrt{5^2 + 22} : 2$;

b. $2 \cdot \sqrt{9} + 3 \cdot \sqrt{16}$

d. $\sqrt{7^2 - 26} : 2$

f. $3\sqrt{0.64} - 5 \cdot \sqrt{1.21}$