Math 5b. Classwork 2.

1. The sum of two natural number is 45. First number gives a remainder of 4 when divided by 12, the second gives a remainder 5 when divided by 12. What are these numbers?

2. The sum of two natural number is 54. First number gives a remainder of 11 when divided by 17, the second gives a remainder 9 when divided by 17. What are these numbers?

3. The sum of two natural number is 48. First number gives a remainder of 14 when divided by 19, the second gives a remainder 15 when divided by 19. What are these numbers?

4. The sum of two numbers is 242, and when the larger of these numbers is divided by the smaller one, the quotient is 4, and the remainder is 22. Find the smaller of these numbers.

5. The number *a* is even. Can the remainder of the division of the number a by 6 be equal to 1? 3?

6. When dividing a natural number, *a* by 2, the remainder is 1, and when dividing it by 3, the remainder is 2. What will be the remainder when *a* is divided by 6?

Exponent.

Exponentiation is a mathematical operation, written as a^n, involving two numbers, the base *a* and the exponent *n*. When *n* is a positive integer, exponentiation corresponds to repeated multiplication of the base. In other words, a^n is the product of multiplying *n* bases:

$$a^n = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n\ times}$$

In that case, a^n is called the *n*-th power of *a*, or *a* raised to the power *n*.
The exponent indicates how many copies of the base are multiplied together.

Properties of exponent.

Based on the definition of the exponent, a few properties can be derived.

$$a^n \cdot a^m = \underbrace{a \cdot a \dots \cdot a}_{n\ times} \cdot \underbrace{a \cdot a \dots \cdot a}_{m\ times} = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n+m\ times} = a^{n+m}$$

$$(a^n)^m = \underbrace{a^n \cdot a^n \cdot \dots \cdot a^n}_{m\ times} = \underbrace{\underbrace{a \cdot a \cdot \dots \cdot a}_{n\ times} \cdot \dots \cdot \underbrace{a \cdot a \cdot \dots \cdot a}_{n\ times}}_{m\ times} = a^{n \cdot m}$$

If a number *a* in a power on *n* multiplied by the number *a* one more time, the total number of multiplied bases increased by 1:

$$a^n \cdot a = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times} \cdot a = \underbrace{a \cdot a \cdot a \cdot a \ldots \cdot a}_{n+1\ times} = a^{n+1} = a^n \cdot a^1$$

In order to have the set of power properties consistent, any number in the first power is the number itself. In other words, $a^1 = a$ for any number a.

Also, a^n can be multiplied by 1:

$a^n = a^n \cdot 1 = a^{n+0} = a^n \cdot a^0$

In order to have the set of properties of exponent consistent, $a^0 = 1$ for any number a, but 0.

If there are two numbers a and b:

$$(a \cdot b)^n = \underbrace{(a \cdot b) \cdot \ldots \cdot (a \cdot b)}_{n\ times} = \underbrace{a \cdot \ldots \cdot a}_{n\ times} \cdot \underbrace{b \cdot \ldots \cdot b}_{n\ times} = a^n \cdot b^n$$

All these properties can be summarized:

1. $a^n = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times}$
2. $a^n \cdot a^m = a^{n+m}$
3. $(a^n)^m = a^{n \cdot m}$
4. $a^1 = a$, for any a
5. $a^0 = 1$, for any $a \neq 0$
6. $(a \cdot b)^n = a^n \cdot b^n$

Positive and negative numbers:

- A positive number raised into any power will result a positive number.
- A negative number, raised in a power, represented by an even number is positive, represented by an odd number is negative.

If a number a in a power n is divided by the same number in a power m,

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}} = \left(\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}\right) : \left(\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}\right) = a^n : a^m = a^{n-m}$$

So, we can say that

$$a^{-1} = \frac{1}{a^1} = \frac{1}{a}; \qquad a^{-n} = \frac{1}{a^n};$$

Let's see how our decimal system of writing numbers works when we use the concept of exponent:

$3456 = 1000 \cdot 3 + 100 \cdot 4 + 10 \cdot 5 + 1 \cdot 6 = 10^3 \cdot 3 + 10^2 \cdot 4 + 10^1 \cdot 5 + 10^0 \cdot 6$

The value of a place of a digit is defined by a power of 10 multiplied by the digit. Very large numbers can be written using this system, as well as very small numbers.

$$0.3 = \frac{1}{10} \cdot 3 = \frac{1}{10^1} = 10^{-1} \cdot 3$$

$$24.345 = 10^1 \cdot 2 + 10^0 \cdot 4 + 10^{-1} \cdot 3 + 10^{-2} \cdot 4 + 10^{-3} \cdot 5$$

Scientists work with very large and very small things, from galaxies to viruses. They need to be able to write numbers, describing the object of interest, for example the distance between two galaxies or the diameter of a virus.

One of the most important numbers in the universe is the speed of light.

299 792 458 m / s. It's very convenient to represent it as a decimal starting with units and multiplied by a power of 10.

$$299\ 792\ 458\ m\ per\ s = 2.99792458 \cdot 10^8\ m\ p\ s.$$

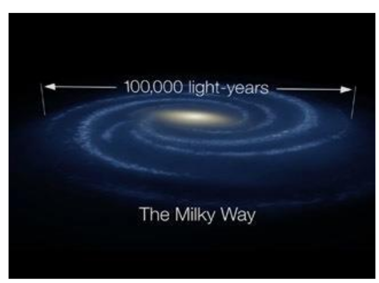
Let's convert the value to kilometers per hour. Each kilometer is 1000 meters, so we need to divide it by 1000:

$$3 \cdot \frac{10^8}{10^3} mps = 3 \cdot 10^{8-5} = 3 \cdot 10^5 km\ per\ s$$

In each hour there are 3600 seconds, or $3.6 \cdot 10^3$ seconds. To find out the speed of light in km per hour we now need to multiply the speed in seconds by $3.6 \cdot 10^3$

$$3 \cdot \frac{10^8}{10^3} mps = 3 \cdot 10^{8-5} = 3 \cdot 10^5 km\ per\ s = 3 \cdot 10^5 \cdot 3.6 \cdot 10^3 = 10.8 \cdot 10^8\ km\ p\ h$$

The Milky Way galaxy has a diameter of 105,700 light years, so the light will travel from one end to the other through its center in 105700 years.

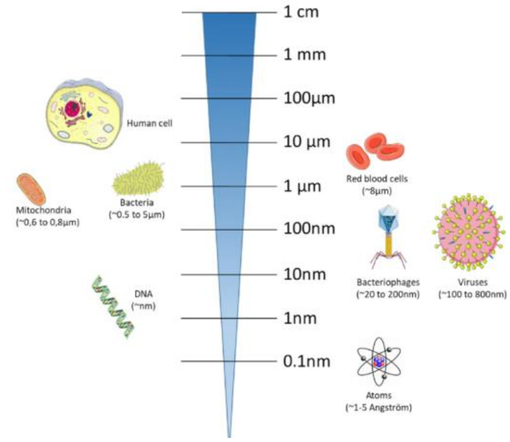


How far is one side from the other in the Milky Way in kilometers? 10.8·108$km\ p\ h$ ·105700 $years$.

How many hours in a year? $24 \cdot 365.25 = 8766 \approx 8.8 \cdot 10^3\ hours$

$$10.8 \cdot 10^8\ km\ p\ h\ \cdot 105700\ years \approx 1.08 \cdot 10^9\ km\ p\ h \cdot 8.8 \cdot 10^3 \cdot 1.06 \cdot 10^5 \approx 10 \cdot 10^{17} km$$
$$\approx 10^{18} km.$$

This way to write numbers is called scientific notation, it's used a lot in science for describing various objects, big and small. Let's take a look on the small things, like bacteria and viruses.



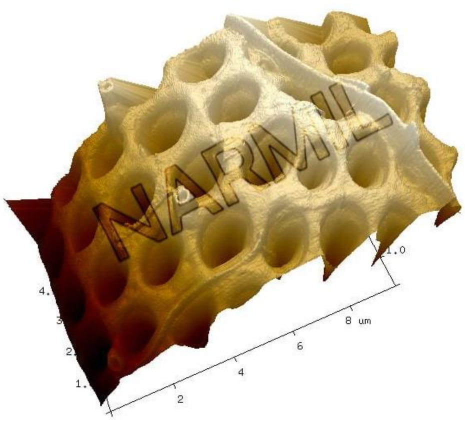
$$1cm = 0.01m = \frac{1}{100}m = \frac{1}{10^2}m = 10^{-2}m$$

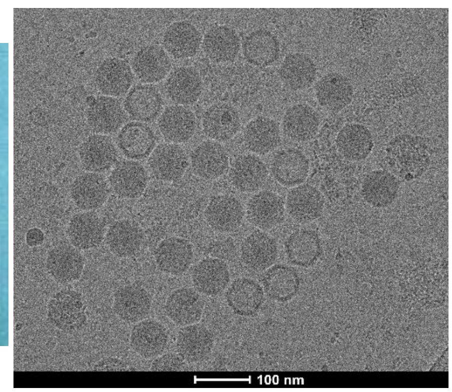
$$1mm = 0.001m = \frac{1}{1000}m = \frac{1}{10^3}m = 10^{-3}m$$

$$1\mu m = 10^{-6}m, \quad 1nm = 10^{-9}m$$

Bactria are between $0.5 - 1.5\ \mu m$

$$0.5\mu m = 0.5 \cdot 10^{-6}m = 5 \cdot 10^{7}m$$





Prefix	Symbol for Prefix		Scientific Notation
exa	E	1 000 000 000 000 000 000	10^{18}
peta	P	1 000 000 000 000 000	10^{15}
tera	T	1 000 000 000 000	10^{12}
giga	G	1 000 000 000	10^{9}
mega	M	1 000 000	10^{6}
kilo	k	1 000	10^{3}
hecto	h	100	10^{2}
deka	da	10	10^{1}
----	--	1	10^{0}
deci	d	0.1	10^{-1}
centi	c	0.01	10^{-2}
milli	m	0.001	10^{-3}
micro	µ	0.000 001	10^{-6}
nano	n	0.000 000 001	10^{-9}
pico	p	0.000 000 000 001	10^{-12}
femto	f	0.000 000 000 000 001	10^{-15}
atto	a	0.000 000 000 000 000 001	10^{-18}

Exercises:

1. Continue the sequence:

   $a.\ 1, 4, 9, 16 \dots$ $\quad b. 1, 8, 27, \dots$ $\quad c.\ 1, 4, 8, 16 \dots$ $\quad d.\ 1, 3, 9, 27 \dots$

2. Write the following products as exponents:

   Example:

   $-2 \cdot 2 \cdot 2 \cdot 2 = -2^4;$ $\quad (-2) \cdot (-2) \cdot (-2) \cdot (-2) = (-2)^4$

   $a.\ (-3) \cdot (-3) \cdot (-3) \cdot (-3);$ $\quad b.\ (-5m)(-5m) \cdot 2n \cdot 2n \cdot 2n;$

   $c.\ -3 \cdot 3 \cdot 3 \cdot 3;$ $\quad d.\ -5m \cdot m \cdot 2n \cdot n \cdot n;$

   $e.\ (ab) \cdot (ab) \cdot (ab) \cdot (ab) \cdot (ab) \cdot (ab);$ $\quad f.\ -q \cdot q \cdot q \cdot q \cdot q;$

   $g.\ 4 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7;$ $\quad h.\ (p-q) \cdot (p-q) \cdot (p-q);$

3. What digits should be put instead of * to get true equality? How many solutions does each problem have?

   $a.\ (2*)^2 = **1;$ $\quad b.\ (3*)^2 = ***6$ $\quad c.\ (7*)^2 = ***5$ $\quad d.\ (2*)^2 = **9$

4. Without doing calculations, prove that the following inequalities hold:

   Example:

   $39^2 < 2000$: $\quad 39 < 40,$ $\quad 39^2 < 40^2 = 1600;$ $\quad 1600 < 2000.$

   $a.\ 29^2 < 1000;$ $\quad b.\ 48^2 < 3000;$ $\quad c.\ 42^2 > 1500;$ $\quad d. 67^2 > 3500$

5. Evaluate:

   $(-3)^2;$ $\quad -3^2;$ $\quad (-3)^3;$ $\quad 2^7;$ $\quad (-2)^7;$ $\quad -2^7;$ $\quad (2 \cdot 3)^3;$ $\quad 2 \cdot 3^3;$ $\quad \left(\frac{1}{3}\right)^2;$ $\quad \frac{1}{3^2};$

6. Let's take a look on a sequence of numbers:

   $2,$ $\quad 2^2,$ $\quad 2^3,$ $\quad 2^4, \dots$ $\quad 2^{12}$

   What are the last digits of this numbers? Can you tell the last digit of 2^{13}, $\quad 2^{14}$, $\quad 2^{15}$?

   What about 2^{32}, $\quad 2^{49}$, $\quad 2^{62}$?

   Can you tell what would be the last digit of 2022^{23}? 2025^{23}? 2023^{23}? 2026^{23}

7. Compare the following exponents:

   a. 2^{10} *and* 10^3; *b.* 10^{100} *and* 100^{10}

   c. 2^{300} *and* 200; *d.* 31^{16} *and* 17^{20}; *e.* 4^{53} *and* 15^{45}

8. Prove that
   $8^5 + 2^{11}$ is divisible by 17
   $9^7 - 3^{10}$ is divisible by 20

9. $x^5 < y^8 < y^3 < x^6$
   Where 0 should be placed?