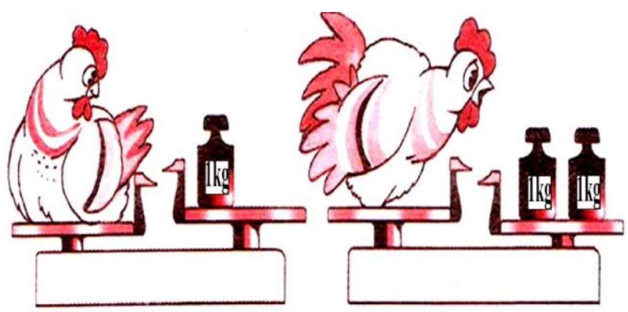
a. A sequence of four numbers was written, each of which is 3 times larger than the previous one. The last number is 486. Find the first number.
b. A sequence of four numbers was written, each of which is 6 times smaller than the previous one. The last number is 2. Find the first number.

30. Find the sum $(a+b)+(a+c)+(a+a)$, if it is known that $a+b+c=8$

Chapter 9.
Decimals

In a process of measurement, we compare a standard unit, such as 1m for length, 1kg for mass, 1degree Celsius for temperature, and so on (we can use another standard units, for example 1 foot, 1 degree Fahrenheit) with the quantity we are measuring. It is very likely that our measurement will not be exact and whole

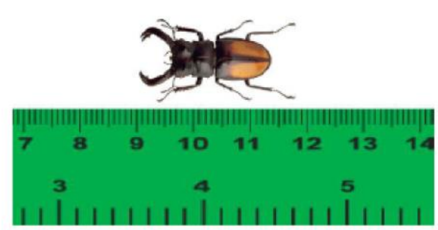


number of standard units will be either smaller, or greater than the measured quantity. In order to carry out more accurate measurement, we have to break our standard unit into smaller equal parts. We can do this in many different ways. For example, we can take $\frac{1}{2}$ of a standard unit and continue measuring. If we didn't get exact n units plus $\frac{1}{2}$ of a unit, we have to subdivide further:

$$n+\frac{1}{2}+\frac{1}{2}\cdot\left(\frac{1}{2}\right)+\cdots$$

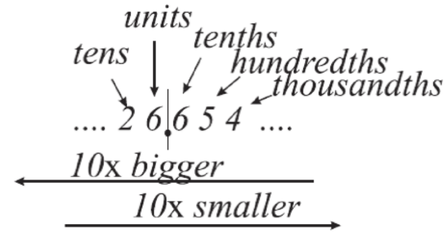
It turns out that perhaps the most convenient way is to divide a unit into 10 equal parts, then each of one tenth into another 10 even smaller equal parts and so on. In this way we will get a series of fractions with denominators 10, 100, 1000 and so on:

$$\frac{1}{10},\frac{1}{100},\frac{1}{1000}\ldots.$$



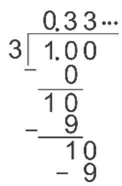
The result of our measurement can be written in a 10 based place value system.

$$26.654 = 10 \cdot 2 + 1 \cdot 6 + \frac{1}{10} \cdot 6 + \frac{1}{100} \cdot 5 + \frac{1}{1000} \cdot 4$$
$$= 10 \cdot 2 + 1 \cdot 6 + \frac{6}{10} \cdot + \frac{5}{100} + \frac{4}{1000}$$
$$= 10 \cdot 2 + 1 \cdot 6 + \frac{600}{1000} + \frac{50}{1000} + \frac{4}{1000}$$



Of course, all such numbers can be expressed in the fractional notation as fractions with denominators 10, 100, 1000 …, but in decimal notation all arithmetic operations are much easier to perform.

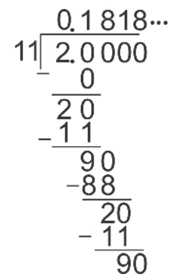
How the fraction can be represented as decimal? One way to do it, just divide numerator by denominator, as usual. For example:



$$\frac{1}{3} = 1:3 = 0.3333 \ldots = 0.\underline{3}$$

Another example,

$$\frac{2}{11} = 2:11 = 0.1818 \ldots = 0.\underline{18}$$



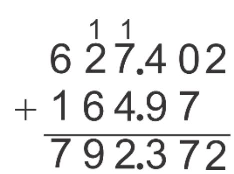
$$\frac{3}{5} = 3:5 = 0.6$$

Can you notice the difference? If the denominator of the fraction can be prime factorized into the product of only 2 and/or 5, the fraction can be written as a fraction with denominator 10, 100, 1000 … Such fraction can be represented as a finite decimal, any other fraction will be written as infinite periodical decimal. For now, we are going to work only with finite decimals.

Examples:

$$0.3 = \frac{3}{10}; \quad 0.27 = \frac{2}{10} + \frac{7}{100} = \frac{27}{100}; \quad 0.75 = \frac{75}{100} = \frac{3 \cdot 25}{4 \cdot 25} = \frac{3}{4}$$
$$\frac{1}{25} = \frac{1}{5 \cdot 5} = \frac{1 \cdot 2 \cdot 2}{5 \cdot 5 \cdot 2 \cdot 2} = \frac{4}{10 \cdot 10} = \frac{4}{100} = 0.04$$
$$\frac{7}{8} = \frac{7}{2 \cdot 2 \cdot 2} = \frac{7 \cdot 5 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} = \frac{875}{1000} = 0.875$$

As you probably already noticed, the decimal (based on 10) system of writing numbers is very consistent, and we can write very big and very small numbers in a very similar way. This system is very convenient when we perform the arithmetic calculations. Addition and subtraction can be completed by exactly the same way as with natural numbers, decimal point should be placed one on the top of the other.



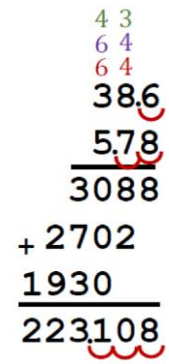
How do we perform the multiplication? First, let’s see why we need to write 0 on the right side of the number when multiplying it by 10:

$$245 \cdot 10 = (100 \cdot 2 + 10 \cdot 4 + 5) \cdot 10 = 100 \cdot 10 \cdot 2 + 10 \cdot 10 \cdot 4 + 10 \cdot 5$$
$$= 1000 \cdot 2 + 100 \cdot 4 + 10 \cdot 5 + 1 \cdot 0 = 2450$$

By multiplying the number by 10 we changed the place values for all digits and added one more place for units.

$$245 \cdot 100 = (100 \cdot 2 + 10 \cdot 4 + 5) \cdot 100 = 100 \cdot 100 \cdot 2 + 10 \cdot 100 \cdot 4 + 100 \cdot 5$$
$$= 10000 \cdot 2 + 1000 \cdot 4 + 100 \cdot 5 + 10 \cdot 0 + 1 \cdot 0 = 24500$$

If we need to multiply the decimal by 10 (or 100)
Using the distributive property, we proved that the result will be the number with decimal point moved one step to the right. (2 steps for multiplication by 100, and so on), It's equivalent to increasing all place values 10 times.



Dividing by 10, we're just multiplying by $\frac{1}{10}$.

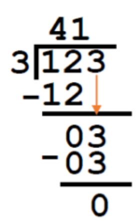
$$230{:}\,10 = 230 \cdot \frac{1}{10} = (100 \cdot 2 + 10 \cdot 3 + 1 \cdot 0) \cdot \frac{1}{10} = \frac{100}{10} \cdot 2 + \frac{10}{10} \cdot 3 + \frac{0}{10} = 20 + 3$$
$$= 23$$

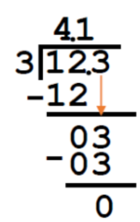
$$235{:}\,10 = 235 \cdot \frac{1}{10} = (100 \cdot 2 + 10 \cdot 3 + 1 \cdot 5) \cdot \frac{1}{10} = \frac{100}{10} \cdot 2 + \frac{10}{10} \cdot 3 + \frac{1}{10} \cdot 5$$
$$= 20 + 3 + \frac{5}{10} = 23.5$$

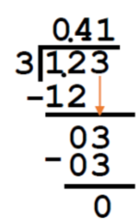
As the result, we are reducing values of each place 10 times, and we have to move the decimal point one step to the left.
To perform the long multiplication of the decimals, we do the multiplication procedure as we would do with natural numbers, regardless the position of decimal points, then the decimal point should be placed on the resulting line as many steps from the right side as the sum of decimal digits of both numbers. When we did the multiplication, we didn't take into the consideration the fact, that we are working with decimals, it is equivalent to the multiplication of each number by 10 or 100 or 1000 … (depends on how many decimal digits it has). So, the result we got is greater by $10 \cdot 100 = 1000$ (in our example) time than the one we are looking for:

$$38.6 \cdot 5.78 = 38.6 \cdot 10 \cdot 5.78 \cdot 100{:}\,(10 \cdot 100) = 386 \cdot 578{:}\,1000$$

Division. To do the long division of the decimal by a hole number, do it as usual, but when the first digit after the decimal point is going to be placed down, put the decimal point into the answer.







If the divisor is decimal as well, it needs to be multiplied by 10(100, 1000…), depends on how many digits it has after decimal point; the dividend also should be multiplied by the same number to get the correct answer. (If we are dividing by a number which is 10 times larger, the number to be divided is also should be 10 times bigger. For example, to divide 123.452 by 1.23 we need to multiply 1.23 by 100:

$$1.23 \cdot 100 = 123$$

to get a whole number. Then multiply123.452 by 100 too.

$$123.452 \cdot 100 = 12345.2$$

Examples:
Fraction to decimals:

$$\frac{2}{10} = 0.2; \qquad \frac{3}{8} = \frac{3}{2\cdot2\cdot2} = \frac{3\cdot5\cdot5\cdot5}{2\cdot2\cdot2\cdot5\cdot5\cdot5} = \frac{3\cdot125}{10\cdot10\cdot10} = \frac{375}{1000} = 0.375;$$

$$\frac{7}{25} = \frac{7\cdot4}{25\cdot4} = \frac{28}{100} = 0.25;$$

Which fractions can be represented as a finite decimal?

$$\frac{3}{4} = \frac{3}{2\cdot2}\ can\ be\ represented;$$

$$\frac{5}{6} = \frac{5}{3\cdot2} - can't\ be\ represented, has\ 3\ as\ a\ factor.$$

Exercises:

1. Write in decimal notation the following fractions:

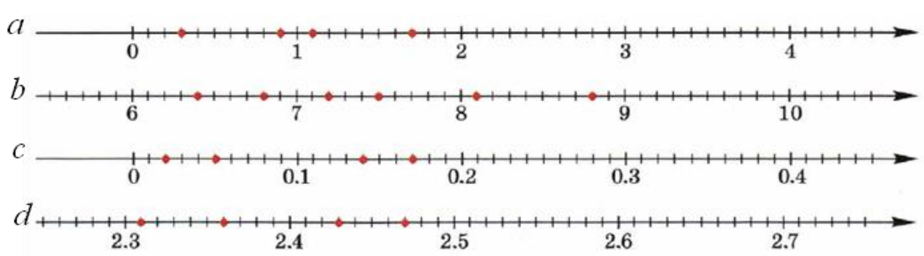
   Example:

$$1\frac{3}{25} = 1 + \frac{3}{25} = 1 + \frac{3\cdot4}{25\cdot4} = 1 + \frac{12}{100} = 1.12$$

$$a.\ 1\frac{1}{10}; \quad 2\frac{4}{10}; \quad 4\frac{9}{10}; \quad 24\frac{25}{100}; \quad 98\frac{3}{100};$$

$$b.\ 1\frac{1}{100}; \quad 4\frac{333}{1000}; \quad 8\frac{45}{1000}; \quad 75\frac{8}{10000}; \quad 9\frac{565}{10000}$$

2. Which numbers are marked on the number lines:



3. Evaluate:

$$a.\ 1.2 + 2.3 + 3.4 + 4.5 + 5.6 + 6.7 + 7.8;$$

$$b.\ 2.3 + 3.4 + 4.5 - 5.6 + 6.7 + 7.8 + 8.5 + 9.2;$$

$$c.\ 1.7 + 3.3 + 7.72 + 3.28 + 1.11 + 8.89;$$

$$d.\ 18.8 + 19 + 12.2 + 11.4 + 0.6 + 11;$$

4. On a graph paper draw a number line, use 10 squares as a unit. Mark points with coordinates 0.1, 0.5, 0.7, 1.2, 1.3, 1.9.

5. Write the numbers in an extended form;

   Example:

$$312.23 = 100\cdot3 + 10\cdot1 + 1\cdot2 + \frac{1}{10}\cdot2 + \frac{1}{100}\cdot3$$

34.2; 231.51; 76.243; 25.34; 0.23; 0.0023

6. Write in decimal notation:

$$\frac{173}{10};\quad \frac{173}{100};\quad \frac{173}{1000};\quad \frac{173}{1000};$$

7. Write in decimal notation:

$a.\ 2\frac{18}{100};\quad 5\frac{3}{100};\quad 1\frac{238}{1000};\quad 8\frac{8}{1000};\qquad b.\ \frac{39}{10};\quad \frac{187}{10};\quad \frac{341}{100};\quad \frac{1002}{1000}$

8. Which fractions below can be written in as a finite decimal:

$$\frac{1}{2},\ \frac{2}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \frac{1}{9},\ \frac{1}{10},$$
$$\frac{1}{11},\ \frac{1}{12},\ \frac{1}{13},\ \frac{1}{14},\ \frac{1}{15},\ \frac{1}{16},\ \frac{1}{17}$$

Why do you think so?

9. Write decimals as fractions and evaluate the following expressions:

$a.\ \frac{2}{3}+0.5;\qquad b.\ \frac{1}{3}\cdot 0.9;\qquad c.\ \frac{3}{16}\cdot 0.16$

$d.\ 0.6-\frac{2;}{5}\qquad e.\ 0.4:\frac{2}{7};\qquad f.\ \frac{9}{20}:0.03$

10. Without performing calculations, for each expression from the first row, find the corresponding equal expression from the second row and write the corresponding equalities.

Example:

$$0.125+\frac{1}{4}=\frac{1}{8}+0.25$$

$\frac{3}{4}-0.5;\qquad \frac{1}{4}-0.2;\qquad \frac{1}{2}-0.125$

$0.5-\frac{1}{8};\qquad 0.75-\frac{1}{2};\qquad 0.25-\frac{1}{5};$

11. Answer:

Which part of 1 m is 1 cm?
Which part of 1 cm is 1 mm?
Which part of 1 kg is 1 g?
Which part of 1 km is 1 m?
Which part of 1 m is 1 dm?
Which part of 1 g is 1 mg?

12. 1 kilogram of candies costs 16 dollars. How much

a. 0.5 kg will cost?
b. 1.2 kg will cost?
c. 0.75 kg will cost?
d. 0.4 kg will cost?
e. 2.5 kg will cost?

13. Represent time in hours, if possible, write the answer as a finite decimal.

$a.\ 1\,h\ 12\,min;\qquad b.\ 2\,h\ 20\,min;\qquad c.\ 10\,h\ \ 45\,min;$

$d.\ 1\,h\ 40\,min;\qquad e.\ 3\,h\ 50\,min;\qquad f.\ 2\,h\ \ 48\,min;$

14. What digit can be placed instead of asterisks, so the expressions are true?
    $a.\ 0.488 < 0.4*8;$ $b.\ 1*93 < 11.93;$ $c.\ 3.07 < 3.0*;$ $d.\ 6.*9 < 6.38$

15. Find the unknown:
    $a.\ 3.3 - 0.3y = 0.33$ $b.\ b:8 - 0.88 = 8.8$

16. In a driving school, a car with an instructor and three students went for a ride. The instructor drove $\frac{2}{5}$ of the total distance plus 5 km, two students each drove $\frac{1}{4}$ of the distance, and the third student drove the remaining 10.5 km. What was the total length of the itinerary?

17. Ancient Greek scientist Aristotle was born in 384 and died in 322. Another Greek scientist, Pythagoras, was born in 570 and died in the year 495. Ancient Greek historian Plutarch was born in 46 and died in 120. Who among them was born earlier? For how long did they live?

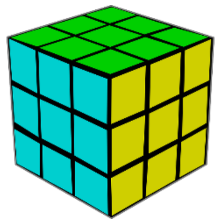
18. Evaluate (answer is 25):

$$\frac{5.6 \cdot 3\frac{1}{3} \cdot 0.63}{4.9 \cdot 0.018 \cdot 5\frac{1}{3}};$$

19. A few kids went to the forest to pick mushrooms. If Anna gives half of her mushrooms to Vita, all the children will have the equal number of mushrooms, if instead Anna gives all her mushrooms to Alex, then Alex will have as many mushrooms as all the other kids combined. How many kids went to the forest for mushrooms?

20. On Halloween night, Peter ate half of the chocolates he had collected. The next day, he ate half of the remaining candies and gave the rest to his younger brother. He gave his brother 5 chocolates. How many candies did Peter collect?

21. A cube is cut into 27 identical smaller cubes by making two cuts parallel to each of the three pairs of cube's faces (similar to Rubik's cube).
    a. How many small cubes will have three faces painted?
    b. How many small cubes will have two faces panted?
    c. How many small cubes will have one face painted?
    d. How many small cubes will not have painted faces at all?



22. The farmer brought a basket of apples to the market. To the first customer, he sold half of all his apples and half an apple more, to the second customer - half of the remainder and half an apple more, to the third - half of the remainder and half an apple more, and so on. However, when the sixth customer came and bought half of the remaining apples and half an apple, it turned out that, like the other buyers, all his apples were whole, and the farmer sold all his apples. How many apples did he bring to the market?

23. Evaluate:

$$\underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{13\ times}+\underbrace{\frac{1}{4}+\frac{1}{4}+\cdots+\frac{1}{4}}_{7\ times}-\underbrace{\frac{1}{4}+\frac{1}{4}+\cdots+\frac{1}{4}}_{25\ times};$$

24. How will the product change if:
    a. one factor is increased 9 times;
    b. one factor is decreased 7 times;
    c. one factor is decreased 2 times, and the other is decreased 8 times;
    d. one factor is increased 4 times, and the other is increased 5 times;
    e. one factor is increased 12 times, and the other is decreased 4 times;
    f. one factor is increased 3 times, and the other is decreased 6 times;
    g. one factor is increased n times, and the other is increased 2 times;
    h. one factor is decreased t times, and the other is decreased 3 times?

25. *252 students from the school are going on a field trip. Several identical buses are ordered for them. However, it turned out that if buses with 6 more seats were ordered, one less bus would be needed. How many larger buses need to be ordered if, in both cases, all buses are expected to be filled with no empty seats?

26. There are the same number of apples in each of the five boxes. If 60 apples are removed from each box, the total number of apples left in all the boxes will be equal to the number of apples that were originally in two boxes. How many apples were there in each box?

27. The sum of six different natural numbers is 22. Find these numbers.