Math 4b. Classwork 20.

1. $a^n = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n\ times}$
2. $a^n \cdot a^m = a^{n+m}$
3. $(a^n)^m = a^{n \cdot m}$
4. $a^1 = a,$ for any a
5. $a^0 = 1,$ for any $a \neq 0$
6. $(a \cdot b)^n = a^n \cdot b^n$

- A positive number raised into any power will result a positive number.
- A negative number, raised in a power, represented by an even number is positive, represented by an odd number is negative.

Exponent is very interesting mathematical operation. There is the story of the invention of the game of chess. The king ordered a new game because he was bored by the old games, was so happy about the new chess game that he said to the inventor: "*Name your reward and you will get it!*" The inventor asked for a simple reward. *"I would like to have one grain of rice on the first chess square, two on the second, four on the third and so on, doubling the amount of rice every square.*" The legend says that the King was surprised he didn't ask for gold but was quite content that the inventor asked for so little. But when the court scholars told him there wasn't enough rice in the whole world to fill the chess board, he had to admit his loss:

$1 + 2 + 2^2 + 2^4 + \dots + 2^{63} = 18{,}446{,}744{,}073{,}709{,}551{,}615$

The weight of the rice grain is about 0.03g. so:

$$1.8 \cdot 10^{19} \cdot 0.03 = 5.4 \cdot 10^{17}\ g. or\ about\ 5.4 \cdot 10^{14} kg\ or\ 10^{15} lb.$$

$$1.05 \cdot 1.05 \cdot 1000 = 1.05^2 \cdot 1000 = 1102.5$$

Let's take a look on a fraction $\frac{27}{81}$:

$$\frac{27}{81} = \frac{3^3}{3^4} = \frac{3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3} = \frac{\cancel{3} \cdot \cancel{3} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot 3} = \frac{1}{3} = 3^{3-4} = 3^{-1}$$

$$\frac{1}{3} = 3^{-1}; \quad \frac{1}{3^3} = \frac{1}{3 \cdot 3 \cdot 3} = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = 1 : 3^1 : 3^1 : 3^1 = 1 : 3^3 = 3^{-3}$$

Negative power can be seen as a division exactly the same way as multiplication for the positive power.

If a number a in a power n is divided by the same number in a power m,

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \dots \cdot a}_{n\ times}}{\underbrace{a \cdot a \cdot \dots \cdot a}_{m\ times}} = \left(\underbrace{a \cdot a \cdot \dots \cdot a}_{n\ times}\right) : \left(\underbrace{a \cdot a \cdot \dots \cdot a}_{m\ times}\right) = a^n : a^m = a^{n-m}$$

So, we can say that

$$a^{-1} = \frac{1}{a^1} = \frac{1}{a}; \qquad a^{-n} = \frac{1}{a^n};$$

Let's see how our decimal system of writing numbers works when we use the concept of exponent:

$$3456.35 = 1000 \cdot 3 + 100 \cdot 4 + 10 \cdot 5 + 1 \cdot 6 + \frac{1}{10} \cdot 3 + \frac{1}{100} \cdot 5$$
$$= 10^3 \cdot 3 + 10^2 \cdot 4 + 10^1 \cdot 5 + 10^0 \cdot 6 + 10^{-1} \cdot 3 + 10^{-2} \cdot 5$$

Very large numbers can be written using this system, as well as very small numbers.

$$30000000 = 10^7 \cdot 3, \qquad \text{but usually this number is written as } 3 \cdot 10^7$$

Problem 1:
What is the last digit of the number 31^{31}? Answer: The last digit of the number is 1, last digit of the product of any numbers of 1 is also 1.
What is the last digit of the number 2^{15}?

2^1	2	2^7	128
2^2	4	2^8	256
2^3	8	2^9	512
2^4	16	2^{10}	1024
2^5	32	2^{11}	2048
2^6	64	2^{12}	4096

Last digits of these numbers are:

2,	4,	8,	6,	2,	4,	8,	6,	2,	4,	8,	6 ...
1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12 ...

The pattern is clear, 2, 4, 8, 6 are repeating. If the number in question is the last digit of 2^{15}, the exponent 15 has to be divided by 4 (period of the pattern):

$$15:4 = 3R3$$

The pattern of 4 numbers can be fit into 15 three times and remainder is 3, third digit of the pattern will be the last digit of the number, 8. Check:

2^{13}	8192	2^{15}	32768
2^{14}	16384	2^{16}	65536

Problem 2.

What is greater 31^{11} or 17^{14}?

31 is less then 32, so 31^{11} is less than 32^{11}. But 32 is 2^5, therefore

$$32^{11} = (2^5)^{11} = 2^{5\cdot 11} = 2^{55}$$

And finally,

$$31^{11} < 32^{11} = (2^5)^{11} = 2^{5\cdot 11} = 2^{55} \text{ or } 31^{11} < 2^{55}$$

On the other hands, 17 is greater than 16, and $17^{14} > 16^{14}$; but 16 is 2^4, therefore

$$16^{14} = (2^4)^{14} = 2^{4\cdot 14} = 2^{56} \; and \; 17^{14} > 2^{56}$$

Finally, $31^{11} < 2^{55}$ and $17^{14} > 2^{56}$. 2^{55} is less than 2^{56}.

$$31^{11} < 2^{55} < 2^{56} < 17^{14}$$

$$31^{11} < 17^{14}$$

Exercises:

1. Evaluate:

$a.\ 2^5;$ $b.\ (-2)^5;$ $c. -2^5;$ $d.\ 3^4;$ $e.\ (-3)^4;$

$f.\ 0.2^5;$ $g.\ (-0.2)^5;$ $h. -0.2^5;$ $i.\ 0.05^2;$ $k.\ (-0.1)^4;$

2. Write the following expressions in a shorter way replacing product with power:

Examples:

$(-a)\cdot(-a)\cdot(-a)\cdot(-a) = (-a)^4,$ $3m\cdot m\cdot m\cdot 2k\cdot k\cdot k\cdot k = 6m^3k^4$

$a.\ (-y)\cdot(-y)\cdot(-y)\cdot(-y);$

$b.\ (-5m)(-5m)\cdot 2n\cdot 2n\cdot 2n;$

$c.\ -y\cdot y\cdot y\cdot y;$

$d.\ -5m\cdot m\cdot 2n\cdot n\cdot n;$

$e.\ (ab)\cdot(ab)\cdot(ab)\cdot(ab)\cdot(ab)\cdot(ab);$

$f.\ p-q\cdot q\cdot q\cdot q\cdot q;$

$g.\ a\cdot b\cdot b\cdot b\cdot b\cdot b;$

$h.\ (p-q)\cdot(p-q)\cdot(p-q);$

3. Write as a product:

   Example: $(-2)^3 = (-2) \cdot (-2) \cdot (-2)$

   $a.\ (-3)^3;$ $b. -3^3;$ $c.\ (2a)^3;$ $d.\ (-2a)^3;$ $e.\ 2a^3$ $f.\ -2a^3$

4. Write as single power:

   Example: $(7^5)^{10} = 7^{5 \cdot 10} = 7^{50}$

   $a.\ (2^2)^3;$ $b.\ (3^4)^2;$ $c. (3^7)^2;$ $d.\ (5^3)^4;$ $e.\ (10^3)^5;$ $f.\ (7^2)^4;$

5. Simplify the expressions:

   $a.\ 2^4 + 2^4;$ $b.\ 2^m + 2^m;$ $c.\ 2^m \cdot 2^m;$

   $d.\ 3^2 + 3^2 + 3^2;$ $e.\ 3^k + 3^k + 3^k;$ $f.\ 3^k \cdot 3^k \cdot 3^k;$

6. What will be last digit of

   $a.\ 2^{22};$ $b.\ 3^{33};$ $c.\ 4^{44};$ $d.\ 5^{55};$ $e.\ 6^{66};$ $f.\ 7^{77};$

7. Compare:
   a. 127^{23} and 513^{18}
   b. 9997^{10} and 100003^8
   c. 5^{300} and 3^{500}