MATH 10
ASSIGNMENT 19: EULER'S FORMULA
MARCH 9, 2025

Euler's formula

Recall the series from last time:

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We have discussed that it has the following properties:

1. It converges for any $x \in \mathbb{R}$
2. $E(x)E(y) = E(x+y)$
3. $E(0) = 1$
4. For small values of x, $E(x) \approx 1 + x$

Thus, if we denote

$$e = E(1) = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718281828\ldots$$

then one can show that $E(x) = e^x$ (for integer x, it easily follows from above properties. For other values, this is taken as definition of e^x.)

We can also consider $E(x)$ for complex values of x. In particular, if $x = it$, t is real, then we can show that $|E(it)| = 1$, so $E(it)$ is on the unit circle (see problem 2 below). Moreover, we have the following formula:

Theorem (Euler's formula)**.** *If t is real, then*

$$E(it) = e^{it} = \cos t + i \sin t$$

In particular, $e^{i\pi} = -1$.

Partial proof of this is given in problem 2 below.

Homework

1. For what values of x do the series below converge? [You can use results discussed last time — in particular, the comparison test and the ratio test.]
   (a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$
   (b) $1 + \frac{x^2}{4} + \frac{x^6}{9} + \cdots + \frac{x^{2n}}{n^2} + \ldots$
   (c) $\sum \frac{(-1)^n x^n}{n(n+1)}$

2. Without using Euler formula, prove the following:
   (a) For any complex z, $\overline{E(z)} = E(\overline{z})$.
   (b) For real t, $\overline{E(it)} \cdot E(it) = 1$, so $|E(it)| = 1$.
   (c) Let $\varphi(t) = \arg(E(it))$, where arg is the argument (angle) of a complex number. Prove that $\varphi(0) = 0$, $\varphi(t_1 + t_2) = \varphi(t_1) + \varphi(t_2)$., and for small values of t, $\varphi(t) \approx t$.
   (d) From the above, can you prove that $\varphi(t) = t$? This would show that $E(it)$ is a complex number with magnitude 1 and argument t, i.e.

$$E(it) = \cos(t) + i \sin(t)$$

   which is exactly the Euler's formula.

3. Separating in Euler's formula real and imaginary parts, show that $\sin(x)$, $\cos(x)$ can be written as series of the form $\sum a_n x^n$.

4. Use Euler's formula and identity $E(x)E(y) = E(x+y)$ to get formulas for $\sin(x+y)$, $\cos(x+y)$ in terms of $\sin(x)$, $\sin(y)$, $\cos(x)$, $\cos(y)$.

5. (a) It is known that the function $f(x) = \frac{1}{\cos x}$ can be written as a series $f(x) = 1 + a_2 x^2 + a_4 x^4 + \dots$. Using the formula for $\cos(x)$ from Problem 3, can you find a_2, a_4?

(b) Show that $\tan(x) = x + c_3 x^3 + c_5 x^5 + \dots$. Find c_3, c_5

***6.** It is known that $\sin(x)$ can also be written as the following infinite product:

$$\sin(\pi x) = \pi x \prod_{1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) = \pi x \left(1 - x^2\right)\left(1 - \frac{x^2}{4}\right)\left(1 - \frac{x^2}{9}\right)\dots$$

Comparing it with the series you got in problem 3, find the formula for $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^4}$ (hint: if you open all parentheses to rewrite the product as a sum, what will be the coefficient of x^3? of x^5?)