MATH 10
ASSIGNMENT 18: SERIES 2

MAR 2, 2025

Series

Recall: given a sequence a_n, we define

$$\sum_{i=1}^{\infty} a_i = \lim S_n, \qquad \text{where}$$

$$S_n = a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

(if this limit exists; otherwise we say that the series diverges and expression $\sum_{i=1}^{\infty} a_i$ is meaningless). For example:

$$1 + r + r^2 + \cdots = \sum_{i=0}^{\infty} r^i = \lim \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}, \qquad |r| < 1$$

(this series is called the *geometric series*).

Note that it is quite possible that the sequence a_n converges but the sequence S_n of partial sums does not converge and thus the series $\sum_{i=1}^{\infty} a_i$ diverges!!

In the last HW, we have proved the following facts.

Theorem.

1. *If a series $\sum a_n$ converges, then $\lim a_n = 0$. (Converse is not true: even if $\lim a_n = 0$, the series may diverge).*
2. *If $0 \le a_n \le b_n$, and $\sum b_n$ converges, then $\sum a_n$ also converges.*

In fact, there is a more general result:

Theorem (Comparison test). *If a_n, b_n are sequences such that $b_n \ge 0$, $|a_n| \le b_n$ and the series $\sum_1^{\infty} b_n$ converges, then $\sum_1^{\infty} a_n$ also converges.*

The proof of this result will be given later. Note that it also works if a_n is a complex sequence (but b_n must be real, as we require $b_n > 0$).

Homework

1. A tortoise is moving on the plane starting at the origin and then going 1 unit along the positive direction of x axis; then turning $90°$ to the left and going for 0.9 units, then turning $90°$ to the left and going for $(0.9)^2$ units, then....
   (a) Show that if we consider the plane as the complex plane $\mathbb{C}$, then the position of the tortoise after n steps will be at the point $1 + r + r^2 + \cdots + r^{n-1}$, where $r = 0.9i$.
   (b) Find where the tortoise will end up in the limit, after infinitely many steps.

2. Let a_n be a sequence such that $\lim \frac{|a_{n+1}|}{|a_n|} = 0.9$
   (a) Show that for large enough n, we have $|a_n| < C(0.95)^n$ for some constant C
   (b) Deduce from this that the series $\sum a_n$ converges.

3. Let a_n be a sequence such that $r = \lim \frac{|a_{n+1}|}{|a_n|}$ exists.
   (a) Show that if $r > 1$, then $\lim a_n$ does not exists, and therefore $\sum_1^{\infty} a_n$ diverges (compare with Problem 1 from previous HW).
   (b) Prove that if $r < 1$, then the series $\sum_1^{\infty} a_n$ converges. [Hint: modify the argument from the previous problem]
   (c) Give examples showing that if $r = 1$, then series $\sum_1^{\infty} a_n$ may converge or diverge.

   This is known as the *ratio test* for series convergence.

4. Use the ratio test from the previous problem to prove that the series

$$\sum \frac{n}{2^n}$$

converges.

5. Prove that for any $x \in \mathbb{C}$, the series

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges. [Hint: use the ratio test from problem 3.]

6. Let $E(x)$ be as defined in the previous problem. Prove that then $E(x+y) = E(x)E(y)$. [Hint: both sides can be written as "double series" $\sum a_{m,n} x^n y^m$. You can use without a proof that in all the series involved, rearranging the terms in any order will not affect the value of the series.]

7. Let $e = \sum_{n=0}^{\infty} \frac{1}{n!} = E(1)$ (we have seen it in the previous homework). Prove that then $E(x) = e^x$:

(a) For all integer x

(b) For all rational $x = p/q$

*(c) For all real x