MATH 10
ASSIGNMENT 17: SERIES

FEB 9, 2025

Series

Given a sequence a_n, consider a new sequence

$$
\begin{aligned}
&S_1 = a_1\\
&S_2 = a_1 + a_2\\
&S_3 = a_1 + a_2 + a_3\\
&\dots\\
&S_n = a_1 + \dots + a_n = \sum_{i=1}^{n} a_i
\end{aligned}
$$

If the sequence $S_1, \dots, S_n$ has a limit, we will write

$$\sum_{i=1}^{\infty} a_i = \lim S_n$$

and call it the sum of the infinite series. In such a situation we say that the infinite series $\sum_1^\infty a_n$ converges.

For example:

$$1 + r + r^2 + \dots = \sum_{i=0}^{\infty} r^i = \frac{1}{1-r}, \qquad |r| < 1$$

Note that it is quite possible that the **sequence** a_n converges but the **series** $\sum_1^\infty a_n$ does not converge!

1. Prove that if the series $\sum_1^\infty a_n$ converges, i.e. the limit $\lim S_n$ exists, then $\lim a_n = 0$. [Hint: $a_n = S_n - S_{n-1}$.]

Note that converse is false — see problem 4 below.

2. Prove that if $0 \le a_n \le b_n$, then

(a) $\sum_{i=1}^{n} a_i \le \sum_{i=1}^{n} b_i$

(b) If the series $\sum_{i=1}^{\infty} b_i$ converges: $\sum_{i=1}^{\infty} b_i = B$, then the series $\sum_{i=1}^{\infty} a_i$ also converges, and $\sum_{i=1}^{\infty} a_i \le B$. [Hint: show that $S_n = \sum_{i=1}^{n} a_i$ is a bounded increasing sequence.]

Note: it is known that a more general fact holds: if $b_i \ge 0$, the series $\sum b_i$ converges, and the sequence a_i is such that $|a_i| \le b_i$, then $\sum a_i$ also converges, even without the assumption that $a_i \ge 0$. However, the proof is much more complicated.

3.

(a) Prove that the series $\sum \frac{1}{n(n+1)}$ converges and find the sum.

(b) Use the previous problem to prove that the series $\sum \frac{1}{n^2}$ converges.

[This problem is essentially a repetition of the last problem in the previous HW.]

4. Prove that the harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

does not converge. Hint: group the terms as follows:

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) \dots$$

and show that the sum of terms inside each parentheses is $\ge 1/2$.

5. Prove that the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

converges, by noticing that $\frac{1}{n!} \le \frac{1}{2^{n-1}}$.

The value of this series is denoted by letter e and is at least as important in math as the number π:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718281828\dots$$

(where we use the convention $0! = 1$)