MATH 10
ASSIGNMENT 13: OPEN AND CLOSED SETS
JAN 5, 2025

Definition 1. A metric space is a set X with a distance function: for any $x, y \in X$ we have a real number $d(x, y)$ such that

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$ for any x, y, and $d(x, y) = 0$ if and only if $x = y$
3. Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

Usual examples are $\mathbb{R}$ (with distance given by $|x_1 - x_2|$), $\mathbb{R}^2$ (with distance given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$) ..., but there are other examples as well.

Given a point $x \in X$ and a positive real number ε, we define ε-neighborhood of x by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

If $S \subset X$, denote by S' the complement of S. Then, for any $x \in X$, we can have one of three possibilities:

1. There is a neighborhood $B_\varepsilon(x)$ which is completely inside S (in particular, this implies that $x \in S$). Such points are called *interior points* of S; set of interior points is denoted by $\text{Int}(S)$.
2. There is a neighborhood $B_\varepsilon(x)$ which is completely inside S' (in particular, this implies that $x \in S'$). Thus, $x \in \text{Int}(S')$.
3. Any neighborhood of x contains points from S and points from S' (in this case, we could have $x \in S$ or $x \in S'$). Set of such points is called the *boundary* of S and denoted ∂S. (Note that it is immediate that $\partial S = \partial S'$)

Thus, the whole space X is union of three pieces:

$$X = \text{Int}(S) \cup \partial S \cup \text{Int}(S')$$

Set S itself contains all of $\text{Int}(S)$ and some (possibly none) points from ∂S.

Definition 2. A set S is called *open* if every point $x \in S$ is an interior point: $S = \text{Int}(S)$ (thus, S contains no points from ∂S).

A set S is called closed if S contains all points from ∂S, i.e. $\partial S \subset S$.

Homework

1. Show that set $\mathbb{R}^2$ with distance defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

is a metric space. (This distance is sometimes called *Manhattan* or taxicab distance — can you guess why?)

2. For each of the following subsets of $\mathbb{R}$, find its interior and boundary and determine if it is open, closed, or neither.
   (a) Set $\mathbb{N} = \{1, 2, 3, \dots\}$.
   (b) Interval $[0, 1]$
   (c) Open interval $(0, 1)$
   (d) Interval $[0, 1)$.
   (e) Set of all rational numbers
   (f) Set consisting of just two points $\{0, 1\}$
   *(g) Set $x^3 + 2x + 1 > 0$
   Are there any subsets of $\mathbb{R}$ which are both open and closed?

3. Let X be an arbitrary metric space, and let $p \in X$. For each of the following subsets of X, find its interior and boundary, and determine if it is open, closed, or neither.
   (a) Open ball $B_r(p) = \{x \mid d(x, p) < r\}$.
   (b) Closed ball $\overline{B}_r(p) = \{x \mid d(x, p) \leq r\}$.
   [Hint: if it makes it easier, consider first $X = \mathbb{R}^2$.]

4. Show that a set S is open if and only if its complement S' is closed.

5. Show that union and intersection of two open sets is open. Is it true if instead of two sets we consider any collection (possibly infinite) of open sets?

Same question about closed sets.

***6.** For a set S, let $\overline{S} = S \cup \partial S = \{x \mid \text{ In any neighborhood of } x, \text{ there are elements of } S\}$. This set is called the *c*losure of S; motivation for this name will become clear in a second.

(a) Show that S is closed if and only if $\overline{S} = S$.

(b) Consider the set $\overline{\overline{S}}$ (note two bars) – closure of closure of S. Show that for any point $p \in \overline{\overline{S}}$, there is a point from S within distance 1 from p.

(c) Show that $\overline{\overline{S}} = \overline{S}$; deduce from it that $\overline{S}$ is closed. (This explains the name closure)

7. Consider the set $\mathbb{R}[x]$ of all polynomials with real coefficients. We take it for granted that every polynomial $p(x)$ has a maximum on interval $[0,1]$, which we denote $\max_{[0,1]} p(x)$, and similarly for minimum.

Define the distance between two polynomials f, g by

$$d(f,g) = \max_{[0,1]} |f(x) - g(x)|.$$

Show that this turns $\mathbb{R}[x]$ into a metric space.