MATH 10
ASSIGNMENT 4: VECTORS AND COORDINATES
OCT 6, 2024

Vectors

A **vector** is a directed segment. We denote the vector from A to B by $\overrightarrow{AB}$. We will also frequently use lower-case letters for vectors: $\vec{v}$.

We will consider two vectors to be the same if they have the same length and direction; this happens exactly when these two vectors form two opposite sides of a parallelogram. Using this, we can write any vector $\vec{v}$ as a vector with tail at given point A. We will sometimes write $A + \vec{v}$ for the head of such a vector.

Vectors are used in many places. For example, many physical quantities (velocities, forces, etc) are naturally described by vectors.

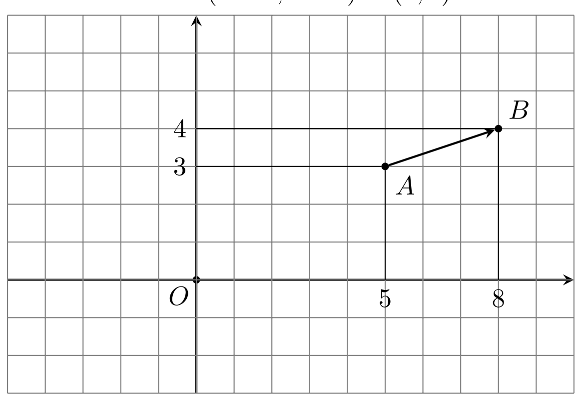
Vectors in coordinates

Recall that every point in the plane can be described by a pair of numbers – its coordinates. Similarly, any vector can be described by two numbers, its x-coordinate and y-coordinate: for a vector $\overrightarrow{AB}$, with tail $A = (x_1, y_1)$ and head $B = (x_2, y_2)$, its coordinates are

$$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$$

For example, on picture below,

$$\overrightarrow{AB} = (8 - 5, 4 - 3) = (3, 1)$$



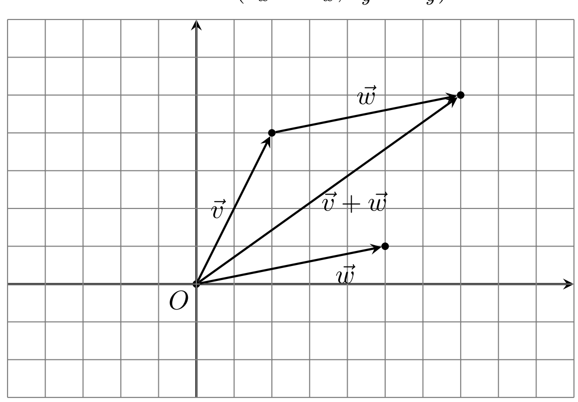
Operations with vectors

Let $\vec{v}$, $\vec{w}$ be two vectors. Then we define a new vector, $\vec{v} + \vec{w}$ as follows: choose A, B, C so that $\vec{v} = \overrightarrow{AB}$, $\vec{w} = \overrightarrow{BC}$; then define

$$\vec{v} + \vec{w} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

In coordinates, it looks very simple: if $\vec{v} = (v_x, v_y)$, $\vec{w} = (w_x, w_y)$, then

$$\vec{v} + \vec{w} = (v_x + w_x, v_y + w_y)$$



Theorem. *So defined addition is commutative and associative:*

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$
$$(\vec{v_1} + \vec{v_2}) + \vec{v_3} = \vec{v_1} + (\vec{v_2} + \vec{v_3})$$

There is no obvious way of multiplying two vectors; however, one can multiply a vector by a number: if $\vec{v} = (v_x, v_y)$ and t is a real number, then we define

$$t\vec{v} = (tv_x, tv_y)$$

Again, we have the usual distributive properties:

$$t \cdot (\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$$
$$(t_1 + t_2)\vec{v} = t_1\vec{v} + t_2\vec{v}.$$

Homework

Unless stated otherwise, we denote by O the origin of the plane.

1. (a) Let $A = (3, 6)$, $B = (5, 2)$. Find the coordinates of the vector $\vec{v} = \overrightarrow{AB}$ and coordinates of the points $A + 2\vec{v}$; $A + \frac{1}{2}\vec{v}$; $A - \vec{v}$.
(b) Repeat part (a) for points $A = (x_1, y_1)$,$B = (x_2, y_2)$

2. Consider a parallelogram $ABCD$ with vertices $A(0, 0)$, $B(3, 6)$, $D(5, -2)$. Find the coordinates of:
(a) vertex C
(b) midpoint of segment BD
(c) Midpoint of segment AC

3. Repeat the previous problem if coordinates of B are (x_1, y_1), and coordinates of D are (x_2, y_2). Use the result to prove that diagonals of a parallelogram bisect each other (i.e., the intersection point is the midpoint of each of them).

4. Recall that the midpoint M of segment AB is defined as the point on the segment such that $|AM| = |MB|$, or equivalently, $\overrightarrow{AM} = \overrightarrow{MB}$.
(a) Without using coordinates, prove that $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$.
(b) Show that if A, B have coordinates $A = (x_1, y_1)$,$B = (x_2, y_2)$, then M has coordinates $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

5. Let AB be a segment, and M a point on the segment which divides it in the proportion 2:1, i.e., $|AM| = 2|MB|$. Show that $\overrightarrow{OM} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OB}$

6. Consider triangle $\triangle ABC$ in the plane.
(a) Let A_1 be the midpoint of segment BC. Write vector $\overrightarrow{OA_1}$ in terms of vectors $\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{OB}$, $\vec{w} = \overrightarrow{OC}$. Repeat the same for midpoint B_1 of segment AC; for midpoint C_1 of segment AB.
(b) Let M_1 be the point on the median AA_1 which divides AA_1 in proportion $2 : 1$. Write vector $\overrightarrow{OM_1}$ in terms of vectors $\vec{u}, \vec{v}, \vec{w}$. Repeat the same for two other medians BB_1 and CC_1.
(c) Show that the three medians intersect at the center of mass of the triangle, which is the point P defined by $\overrightarrow{OP} = \frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$.

7. Let A, B be two distinct points in the plane.
(a) Show that a point X is on the line AB if and only if one can write $X = A + t\,\overrightarrow{AB}$ for some real t.
(b) Show that a point X is on the line AB if and only if $\overrightarrow{OX} = t_1\,\overrightarrow{OA} + t_2\,\overrightarrow{OB}$, where t_1, t_2 are real numbers satisfying $t_1 + t_2 = 1$.
(c) Can you find a similar description for the segment (not line!) AB?

8. Let A, B, C be three distinct points in the plane, not on the same line.
(a) Show that every point X in the plane can be uniquely written as $\overrightarrow{OX} = \overrightarrow{OA} + t\overrightarrow{AB} + s\overrightarrow{AC}$ for some real t, s.
(b) Show that every point X in the plane can be uniquely written as

$$\overrightarrow{OX} = t_1\,\overrightarrow{OA} + t_2\,\overrightarrow{OB} + t_3\,\overrightarrow{AC}$$

for some real t_1, t_2, t_3 satisfying $t_1 + t_2 + t_3 = 1$. (Numbers t_1, t_2, t_3 are called the *baricentric* coordinates).
(c)* What are the conditions on t_1, t_2, t_3 for point X to be inside $\triangle ABC$ (including sides)?