MATH 10
ASSIGNMENT 25: SUBROUPS AND LAGRANGE THEOREM
MAY 4, 2025

Definition. Let G be a group. A subgroup of G is a subset $H \subset G$ which is itself a group, with the same operation as in G. In other words, H must be

1. closed under multiplication: if $h_1, h_2 \in H$, then $h_1 h_2 \in H$
2. contain the group unit e
3. for any element $h \in H$, we have $h^{-1} \in H$.

Examples are given in problem 1 below.
The main result of today is Lagrange theorem:

Theorem. *If G is a finite group, and H is a subgroup, then $|H|$ is a divisor of $|G|$, where $|G|$ is the number of elements in G (also called the order of G).*

The proof of this theorem is given in problem 4 below.

1. Which of the following are subgroups?
(a) $G = \mathbb{Z}$ (with operation of addition), $H = 5\mathbb{Z}$ =multiples of 5.
(b) $G = \mathbb{Z}$ (with operation of addition), $H = \{n = 5k + 1\}$.
(c) $G = S_n$ — permutation group, H =even permutations
(d) $G = S_n$ — permutation group, H =odd permutations
(e) G =all symmetries of regular n-gon, H=all rotations of regular n-gon

2. Let $\mathbb{Z}_n$ be the group of all remainders mod n, with operation of addition (it is commonly called the cyclic group of order n). Identify this group with the group of all rotations of regular n-gon.

3. Describe all subgroups of $\mathbb{Z}$ (hint: any subgroup contains a smallest positive number)

4. Let $H \subset G$ be a subgroup. For any element $g \in G$, define the subset

$$[g] = gH = \{gh, h \in H\}$$

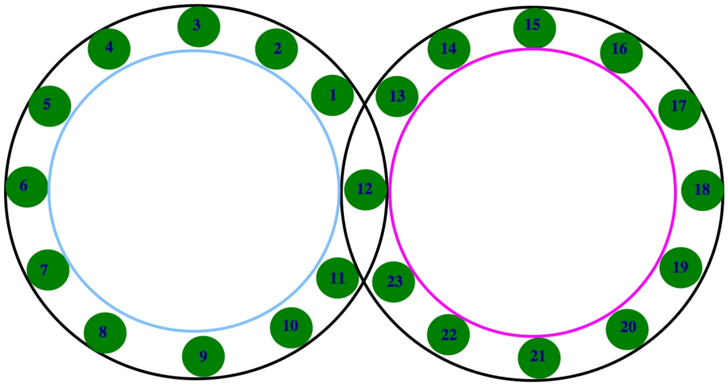
Subsets of this form are called *cosets*. Note that two different elements can define the same coset.
(a) List all cosets in the case when $G = \mathbb{Z}, H = 5\mathbb{Z}$.
(b) Show that two elements x, x' are in the same coset gH iff $x' = xh$ for some $h \in H$.
(c) Show that two cosets g_1H, g_2H either coincide (if $g_1 = g_2h$ for some $h \in H$) or do not intersect at all.
(d) Show that every coset has exactly $|H|$ elements.
(e) Deduce Lagrange theorem:

$$|G| = |H| \cdot (\text{number of cosets})$$

5. In this problem, we consider the permutation group S_n, with $n \geq 5$
(a) Write (12)(34) as a product of cycles of length 3
(b) Show that every even permutation can be written as a product of cycles of length 3.

***6.** Consider the puzzle consisting of 23 numbered balls arranged in two intersecting circles, 12 balls in each, with one ball in common. You can rotate each of the circles by a multiple of 30°. The goal is to have all balls in a correct order. (See figure on next page.)

To solve this puzzle, let $x = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12)$, $y = (12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23)$ be two cycles of length 12 in S_{23}. The question is whether any permutation can be written as a product of these two (and their inverses). Can you answer this question?[Hint: the first step would be computing $xyx^{-1}y^{-1}$].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