Algebraic inequalities

Inequalities come up often when we are trying to *optimize* something - the statement that one choice is better than all other choices is a common type of inequality. *Algebraic* inequalities are important because of the fact that most functions that come up in real life can be approximated by polynomials (in practical applications, it is usually good enough to approximate functions with quadratic polynomials).

In order to prove an algebraic inequality, aside from the usual rules of algebra, you should only need to use a few basic rules:

- positive times (or divided by) positive is positive (and negative times negative is also positive),
- positive plus positive is positive, and
- every square is at least 0.

To disprove an inequality, you just need to find a single example where it is wrong.

Warm ups

1. If two positive (variable) numbers x and y have a fixed (constant) sum s, then experience shows that their product is largest when they are equal to each other (and therefore equal to $s/2$). This suggests the inequality

$$x \cdot y \overset{?}{\leq} \frac{x+y}{2} \cdot \frac{x+y}{2}.$$

   Can you prove or disprove this inequality?

2. If two positive (variable) numbers x and y have a fixed (constant) product p, then experience shows that their sum is smallest when they are equal to each other (and therefore equal to $\sqrt{p}$). This suggests the inequality

$$x, p > 0 \quad \overset{?}{\Longrightarrow} \quad x + \frac{p}{x} \overset{?}{\geq} \sqrt{p} + \sqrt{p}.$$

   Can you prove or disprove this inequality?

3. Suppose that we are trying to minimize the value of the polynomial $x^2 - 6x$, where x is variable. One approach is to find the largest constant c so that we are able to prove the inequality

$$\text{for all } x, \text{ we have } \quad x^2 - 6x \overset{?}{\geq} c.$$

   What is the largest constant c that works, and how can you prove that it works? Which value of x shows that this value of c can't be increased?

Quadratic polynomials

4. We are trying to find the minimum value of the quadratic polynomial

$$Q_1(x, y, z) = x^2 + y^2 + z^2 - 2x + 4y - 6z.$$

After some work, we begin to suspect that the minimum occurs when $(x, y, z) = (1, -2, 3)$, where we have $Q_1(1, -2, 3) = -14$. This suggests that we should attempt to prove or disprove the inequality

$$x^2 + y^2 + z^2 - 2x + 4y - 6z \overset{?}{\geq} -14.$$

5. What is the minimum value of the quadratic polynomial

$$Q_2(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz + 4z?$$

6. Prove or disprove these quadratic inequalities:

   (a) $\frac{1}{2}x^2 + y^2 + 1 \overset{?}{\geq} xy + x$,

   (b) $x^2 + 2xy - 2xz + 2y^2 + 2yz + 6z^2 - z + 1 \overset{?}{\geq} 0$,

   (c) $x^2 - 4xy + 2xz + y^2 - 2yz + 2z^2 \overset{?}{\geq} 0$,

   (d) $x^2 - 2xy - 2xz + y^2 - 2yz + 10z^2 \overset{?}{\geq} 0$,

   (e) $6x^2 - 4xy + 2xz + 3y^2 - 4yz + 2z^2 \overset{?}{\geq} 0$.

   Is there a general procedure for checking whether a quadratic polynomial is ≥ 0?

Sum-of-squares proofs

A good technique for proving polynomial inequalities is to write the difference between both sides as a sum of squares (or of squares times things that you already know are positive).

7. (Power shuffling) Prove the inequality

$$x, y > 0 \quad \overset{?}{\Longrightarrow} \quad x^5 + y^5 \overset{?}{\geq} x^4y + xy^4$$

by writing the difference

$$x^5 + y^5 - x^4y - xy^4$$

as a square times something which is obviously positive.

8. (AM-GM for three variables) Prove the inequality

$$x, y, z > 0 \quad \overset{?}{\Longrightarrow} \quad x^3 + y^3 + z^3 \overset{?}{\geq} 3xyz$$

by writing the difference

$$x^3 + y^3 + z^3 - 3xyz$$

as a sum of squares times something which is obviously positive.

9. (Cauchy-Schwarz) Prove the inequality

$$(a^2+b^2+c^2)(x^2+y^2+z^2) \overset{?}{\geq} (ax+by+cz)^2$$

by writing the difference

$$(a^2+b^2+c^2)(x^2+y^2+z^2) - (ax+by+cz)^2$$

as a sum of squares.

10. Prove that $x^4 + y^4 + z^2 \geq \sqrt{8}xyz$ by writing the difference of both sides as a sum of squares.

11. For which values of the constants a, b, c can the fourth degree polynomial

$$p(x) = x^4 + ax^2 + bx + c$$

be written as a sum of squares? (Hint: introduce a new variable y which satisfies $y = x^2$, and try to rewrite $p(x)$ as a quadratic polynomial $q(x, y)$.)

12. Is it possible to write the polynomial $x^4y^2 + x^2y^4 - 3x^2y^2 + 1$ as a sum of squares?

13. Find a way to write the polynomial

$$(x^2+y^2+1)(x^4y^2+x^2y^4-3x^2y^2+1)$$

as a sum of squares.