PROBABILITIES AND GEOMETRY

APRIL 6, 2025

In most problems in probability, we only consider the situations where the number of possible outcomes is finite, like rolling a die. However, it is also possible to consider experiments where you have infinitely many outcomes such as randomly choosing a point on the interval. Proper mathematical formulation of the probability rules requires some calculus, but for many problems, the following simple rules suffice:

- If a point on a line is chosen at random from some interval I of length L, then the probability that it will be in subinterval $[a, b]$ is proportional to the length of this interval:
$$P(a \le x \le b) = \frac{|b-a|}{L}$$
- If a point on a plane is chosen at random from some region R of area A, then the probability that it will be in subregion R' is proportional to the area of this subregion:
$$P(x \in R') = \frac{A(R')}{A}$$
where $A(R')$ is area of R'.

1. We have a piece of rope of length L which we cut at a random place. What is the average length of the shorter piece?

2. Two people have agreed to meet at certain location between 1pm-2pm. Each one arrives at the location at a random moment between 1 and 2; if the other person is not there, they wait for 15 minutes and then leave.
What is the probability that they will meet?

3. Two numbers are chosen randomly in the interval $[0, 1]$. What is the probability that their sum is larger than $1/2$?

4. Three points are chosen at random on the circle. What is the probability that the triangle formed by these three points is obtuse?

5. Three points A, B, C are chosen at random in the interval $[0, 1]$.
(a) What is the probability that A is the largest?
(b) What is the probability that $A < B < C$?

6. (a) Three points are chosen at random on the circle. What is the probability that the triangle formed by these three points contains the center of the circle?
*(b) Four points are chose at random on the surface of the sphere. What is the probability that the tetrahedron formed by these four points contains the center of the circle?

7. Four points A, B, C, D are chosen on the circle at random. What is the probability that chords AB and CD intersect?

8. In this problem, we talk about probabilities related to choosing a random positive ineteger. Defining it rigorously is difficult, but for this problem you can just use the intuitive understanding without having an exact definition.
(a) If we choose two positive integers m, n at random, independently from each other, what is the probability that 3 is their common prime factor?
(b) What is the probability that they have no common prime factors less than 10?
*(c) What is the probability that m, n are relatively prime? (You might want to look up formula for Euler product)

For the following problem you need to know the following fact:

Average value of function $\sin(x)$ on interval $0 \le x \le \pi$ (we measure angle in radians) is $\frac{2}{\pi}$.

If you are familiar with calculus, you can verify it by computing
$$\frac{1}{\pi}\int_0^\pi \sin\alpha \, d\alpha$$

9. If we place a segment of unit length randomly on the plane, what is the average length (i.e. the expected value) of its projection on the x-axis?

***10.** If we place an $a \times b$ rectangle randomly on the plane, what is the average length of its projection on the x-axis?