MATH CLUB ASSIGNMENT 13: COMBINATORICS

MARCH 23, 2025

1. How many "words" can you get by permuting the letters of word "committee"? (A word is any sequence of letters, not necessarily meaningful.)

2. How many "words" of length 12 can you form using just 2 letters, A and B, if each letter must appear 6 times? What if you are allowed to use 3 letters, A, B, C, each appearing 4 times? Can you get a general formula for number of words using 3 letters, appearing k_1, k_2, k_3 times respectively (thus, total length is $n = k_1 + k_2 + k_3$)?

3. An ant moves along the real line, starting at the origin and each time moving one unit either to the left or to the right. He takes $2n$ steps and ends up again at the origin

(a) Show that the number of such paths is equal to the constant term in the expression $(x+x^{-1})^{2n}$.

(b) Show that this number is equal to ${}_{2n}C_n$.

4. An ant moves in the plane, starting at the origin and each time moving one unit to the left or to the right or up or down. He takes $2n$ steps and ends up again at the origin.

(a) Show that the number of such paths is equal to the constant term in the expression $(x + x^{-1} + y + y^{-1})^{2n}$.

*(b) Prove that this number is equal to $({}_{2n}C_n)^2$. (Hint: rotate the plane 45 degrees. Then each ant's step moves him both horizontally and vertically.)

Stars and Bars

5. How many ways there are to arrange 12 books on 2 bookshelves (top and bottom one)? The order on each bookshelf matters; there are no restrictions on how many of the 12 books are on top shelf.

6. How many solutions does the equation $x_1 + x_2 + x_3 = 2023$ have if x_1, x_2, x_3 must be non-negative integers? what if we require them to be positive integers?

7. How many ways are there to distribute 15 cookies among 6 children? (Cookies are all identical; children are not.)

8. How many different monomials in 3 variables x, y, z of total degree n are there? in 4 variables?

9. How many different monomials in 3 variables x, y, z of total degree n are there if we additionally require that each variable appears with positive degree (i.e. we look for monomials $x^a y^b z^c$, $a > 0$, $b > 0$, $c > 0$, $a + b + c = n$).

10. How many ways there are to put 15 chairs in 4 rooms if every room must have at least one chair? (Chairs are all identical, chairs inside the room are not ordered.)

***11.** How many ways there are to put 15 people in 4 rooms if every room must have at least one person? (People are all different, people inside the room are not ordered.)

Catalan numbers

Consider the sequence of numbers defined by

$$c_0 = 1$$
$$c_1 = c_0c_0 = 1$$
$$c_2 = c_1c_0 + c_0c_1 = 2$$
$$\dots$$
$$c_{k+1} = c_0c_k + c_1c_{k-1} + \cdots + c_kc_0 = \sum_{i=0}^{k} c_ic_{k-i}$$

These numbers are called *Catalan numbers* and appear in many places.

12. Compute first 6 Catalan numbers, up to c_6

13. Consider expression

$$x_1 * x_2 * \cdots * x_{n+1}$$

where $*$ is some binary non-associative operation.

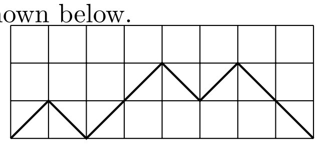
In order to make sense of this expression, we need to insert parentheses to indicate the order of operations. For example, for $n = 2$, there are two ways to do it: $(x_1 * x_2) * x_3$ and $x_1 * (x_2 * x_3)$.

(a) How many ways there are to put parentheses in product of 4 variables $x_1 * x_2 * x_3 * x_4$?

(b) Prove that there are exactly c_n ways to put parentheses in product of $n + 1$ variables [Hint: consider the operation performed last]

14. Prove that for a convex n-gon, there are exactly c_{n-2} ways to draw non-intersecting diagonals which would cut it into triangles. [Hint: choose an edge; look at the triangle containing this edge.]

15. A *Dyck path* is a polyline in the real plane which consists of segments $(1, -1)$ and $(1, 1)$ (i.e., moving diagonally: one unit to the right and one unit either up or down), starts at $(0, 0)$ and ends at $(2n, 0)$ and which never goes below the x-axis (but may touch it). An example of Dyck path with $n = 4$ is shown below.



We will denote the number of all Dyck paths of length $2n$ by D_n.

(a) Show that the number of Dyck paths of length $2n$ which are strictly above the x-axis (except the endpoints) is D_{n-1}.

(b) Show that $D_n = c_n$, i.e. the number of Dyck paths of length $2n$ is the Catalan number c_n (Hint: consider the first time the path touches the x-axis; use this point to divide the path into two subpaths).

(c) Show that the number of Dyck paths is the same as number of sequences of length $2n$, consisting of n symbols $+$ and n symbols $-$ such that in any initial segment of it, there are at least as many $+$ as $-$.

***16.** In this problem, you will prove that the number of Dyck paths of lenght $2n$ (and thus, the Catalan number c_n) is equal to

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

To do it, complete each of the steps below.

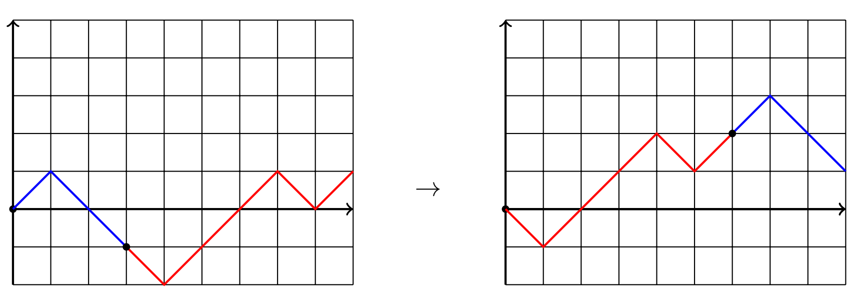
(a) Let S_n be the set of all paths consisting of n segments $(1, -1)$ (diagonally down) and $n + 1$ segments $(1, 1)$ (diagonally up), connecting points $(0, 0)$ and $(2n + 1, 1)$. Show that the number of such paths is $\binom{2n+1}{n}$.

(b) Let us call such a path *positive* if it is strictly above x-axis (except point $(0, 0)$). Show that the number of positive paths is the same as the number of Dyck paths of length $2n$ and thus is equal to the Catalan number c_n.

(c) Consider the following operation on S_n, which we will call *rotation by* k: given an integer k, $0 \le k \le 2n + 1$,

- given a path p, divide into two pieces p_1 (with $0 \le x \le k$) and p_2 (with $k \le x \le 2n + 1$)
- translate p_2 so that it starts at $(0, 0)$
- translate p_1 so that it starts at the endpoint of p_2

The picture below illustrates this operation (for $k = 3$)



Note that for $k = 0$ and $k = 2n + 1$, the rotation does nothing: it leaves the path unchanged. Prove that for every path in S_n, there is exactly one rotation that makes this path positive. [Hint: consider the lowest point on the path.]

(d) Let us group paths in S_n together if one can be obtained from another by some rotation. Prove that then each group has exactly $2n+1$ paths in it, and that in each group, there is exactly one positive path.

(e) Prove that the number of positive paths (and thus, the Catalan number c_n) is given by

$$\frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}$$