Homework for January 21, 2024.

Algebra.

Review the previous classwork handouts. Solve the remaining problems from the previous homework assignments and classwork exercises. Specifically, complete the assignment on inclusion-exclusion principle. Solve the following problems.

1. Construct bijections between the following sets:
    a. (subsets of the set $\{1,\ldots,n\}$) $\leftrightarrow$ (sequences of zeros and ones of length n)
    b. (5-element subsets of $\{1,\ldots,15\}$)$\leftrightarrow$ (10-element subsets of $\{1,\ldots,15\}$)
    c. [set of all ways to put 10 books on two shelves (order on each shelf matters)]$\leftrightarrow$ (set of all ways of writing numbers 1, 2, . . . , 11 in some order) [Hint: use numbers 1. . . 10 for books and 11 to indicate where one shelf ends and the other begins.]
    d. (all integer numbers) $\leftrightarrow$ (all even integer numbers)
    e. (all positive integer numbers) $\leftrightarrow$ (all integer numbers)
    f. (interval (0,1))$\leftrightarrow$ (interval (0,5))
    g. (interval (0,1))$\leftrightarrow$ (halfline $(1,\infty)$) [Hint: try $1/x$.]
    h. (interval (0,1))$\leftrightarrow$ (halfline $(0,\infty)$)
    i. (all positive integer numbers)$\leftrightarrow$ (all integer numbers)
2. Let A be a finite set, with 10 elements. How many bijections f: $A \to A$ are there? What if A has n elements?
3. Let $f\colon \mathbb{Z} \xrightarrow{f} \mathbb{Z}$ be given by $f(n) = n^2$. Is this function injective? surjective?
4. Hotel Infinity is a fictional hotel with infinitely many rooms, numbered 1, 2, 3, ... Each hotel room is single occupancy: only one guest can stay there at any time.
    a. At some moment, Hotel Infinity is full: all rooms are occupied. Yet, when 2 more guests arrive, the hotel manager says he can give rooms to them, by moving some of the current guests around. Can you show how? (Hint: Construct a bijection between the set $\{-1,0,1,2,\ldots\}$ and the set of natural numbers, $\mathbb{N}$).

b. At some moment, Hotel Infinity is full: all rooms are occupied. Still, the management decides to close half of the rooms — all rooms with odd numbers — for renovation. They claim they can house all their guests in the remaining rooms. Can you show how? (Hint: Construct a bijection between the set of all even positive integers $\{2,4,6,\ldots\}$ and $\mathbb{N}$).

c. Next to Hotel infinity, a competitor has built Hotel Infinity 2, with infinitely many rooms numbered by all integers: $\ldots,-2,-1,0,1,2,\ldots$. Yet, the management of the original Hotel Infinity claims that their hotel is no smaller than the competition: they could house all the guests of Hotel Infinity 2 in Hotel Infinity. Could you show how? (Hint: Construct a bijection between the set of all integer numbers $\{\ldots,-2,-1,0,1,2,\ldots\}$ and $\mathbb{N}$).

Bonus recap problems

5. Find the value of the continued fraction given by

$$\{1,2,3,3,3,\ldots\} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cdots}}}}$$

6. Consider the quadratic equation $x^2 = 7x + 1$. Find a continued fraction corresponding to a root of this equation.
7. Using the recurrence relation obtained by solving the old hats problem, derive the formula for a derangement probability,

$$p_n \equiv \frac{!n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

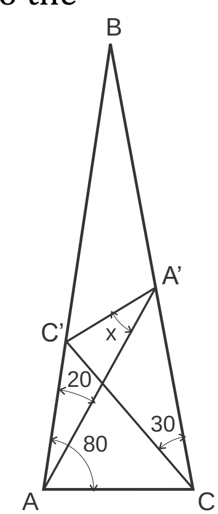
8. Using the inclusion-exclusion principle, find how many natural numbers $n < 1000$ are divisible by 5, 7, 11, or 13.
9. How many passwords of at least 8 characters can one compose using lower- and upper-case letters and numbers 0 to 9?
10. * If 9 dies are rolled, what is the probability that all 6 numbers appear?
11. * How many permutations of the 26 letters of English alphabet do not contain any of the words *pin*, *fork*, or *rope*?

Geometry.

Review the previous classwork notes. Solve the remaining problems from the previous homework (some are repeated below). Solve the following problems.

Problems.

1. Consider all possible configurations of the Apollonius problem (i.e. different possible choices of circles, points and lines). How many possibilities are there? Make the corresponding drawings and write the equations for finding the Apollonius circle in one of them (of your choice).
2. (Skanavi 15.104) Given the right triangle ABC ($\angle C = 90°$), find the locus of points $P(x, y)$ such that $|PA|^2 + |PB|^2 = 2|PC|^2$.
3. (Skanavi 15.109) Points $A(-1,2)$ and $B(4,-2)$ are vertices of the rhombus $ABCD$, while point $M(-2,0)$ belongs to the side CD. Find the coordinates of the vertices C and D.
4. (Skanavi 15.114) Find the circle (write the equation of this circle) passing through the coordinate origin, $O(0,0)$, point $A(1,0)$ and tangent to the circle $x^2 + y^2 = 9$.
5. (Skanavi 15.115) Write the equation of the circle passing through the point $A(2,1)$ and tangent to both X- and Y-axes.
6. In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 30°$ and $\angle BAA' = 20°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.



7. *Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.