MATH 7
HANDOUT 13: QUADRATIC EQUATION

Material covered today

Today we discussed how one solves the quadratic equation:

$$ax^2 + bx + c = 0$$

using two methods: completing the square and using the quadratic formula.

Completing the Square

The first method used is called "completing the square". Here is an example how it works:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 7 = (x+3)^2 - 7 = (x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$$

thus, $x^2 + 6x + 2 = 0$ if and only if $x + 3 + \sqrt{7} = 0$ or $x + 3 - \sqrt{7} = 0$, which gives $x = -3 - \sqrt{7}$, or $x = -3 + \sqrt{7}$.

The same trick works in general: if $a = 1$, then

$$\begin{aligned} x^2 + bx + c = x^2 + 2\tfrac{b}{2}x + c &= \left(x^2 + 2\tfrac{b}{2}x + \tfrac{b^2}{2^2}\right) - \tfrac{b^2}{2^2} + c \\ &= \left(x + \tfrac{b}{2}\right)^2 - \tfrac{b^2 - 4c}{4} = \left(x + \tfrac{b}{2}\right)^2 - \tfrac{D}{4} \end{aligned} \tag{1}$$

where $D = b^2 - 4c$.

Thus, $x^2 + bx + c = 0$ is equivalent to

$$\left(x + \tfrac{b}{2}\right)^2 = \tfrac{D}{4}$$

If a is not equal to 1, the answer is similar: $ax^2 + bx + c = 0$, divide by a, complete the square and the answer is:

$$\left(x + \tfrac{b}{2a}\right)^2 = \tfrac{D}{4a^2}, \qquad D = b^2 - 4ac$$

Therefore, if $D < 0$, there are no solutions; if $D \geq 0$, solutions are

$$\begin{aligned} x + \tfrac{b}{2a} &= \pm\frac{\sqrt{D}}{2a} \\ x &= \frac{-b \pm \sqrt{D}}{2a} \end{aligned} \tag{2}$$

Use the quadratic formula

Given the quadratic equation:

$$ax^2 + bx + c = 0$$

you can use the result from the previous section and use the formula directly to find the two roots:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

The expression that is under the square root is $b^2 - 4ac$. This expression is also called discriminant $D = b^2 - 4ac$.To determine if the quadratic has solutions check the value of the discriminant D:
$D > 0$ - the quadratic equation has 2 solutions
$D = 0$ - the quadratic equation has 1 solution
$D < 0$ - the quadratic equation has no real solutions

Homework

1. Convert the following equations to standard form (open the brackets). Determine the coefficients a, b, and c. Do not solve the equation!

(a) $2(x-3)(x-1)=0$
(b) $(x-2)^2+(2x+3)^2=13-4x$
(c) $(x-4)(x+4)=1$

2. Solve the following quadratic equations by factoring.

(a) $3x^2-2x=0$
(b) $5x^2-16=0$
(c) $x^2-24=1$
(d) $3(x-1)(x+2)=0$

3. Solve the following equations by using the quadratic formula.

(a) $x^2-11x+10=0$
(b) $4y^2-15y+11=0$

4. Complete the square and find the solutions for the following equations:

(a) $x^2+4x+3=0$
(b) $y^2+4y-5=0$

5. Solve the following equations. Carefully write all the steps in your argument.

(a) $x^2-5x+5=0$ (b) $\dfrac{x}{x-2}=x-1$ (c) $x^2=1+x$
(d) $2x(3-x)=1$ (e) $x^3+4x^2-45x=0$

6. The values 3 and 4 satisfy the equation: $x^2+Ax+B=0$, where A and B are constants. What are A and B?

7. If $x+\frac{1}{x}=7$, find $x^2+\frac{1}{x^2}$; $x^3+\frac{1}{x^3}$.