MATH 7
HANDOUT 10: PASCAL'S TRIANGLE

Fundamental Principle of Counting (Multiplication Rule)

If a first task can be performed in **m** ways, and for each of these, a second task can be performed in **n** ways, and for each combination a third task can be performed in **k** ways etc., then this entire sequence of tasks can be performed in $\mathbf{m} \cdot \mathbf{n} \cdot \mathbf{k}$ ways.

Permutations - order matters

Permutations can be used when we choose **k** objects from a set of **n** objects without repetition ("replacement"). For example:

1. Picking first, second and third place winners from a group. There is no replacement as we cannot pick the same person twice, and order matters - Dave, Emma, Page is a different order than Emma, Dave, Page.
If a group has **n** members, then picking 3 can be done in: ${}_nP_3 = n \cdot (n-1) \cdot (n-2)$ ways
The formula for permutations is: ${}_nP_k = \dfrac{n!}{(n-k)!} = n(n-1)(n-2)...(n-k+1)$

2. Permutations of **n** objects: the permutations of all n different objects is **n**!
For example: arranging/ordering all n members of a group can be done in n! ways, listing of favorite desserts in the order of choices: if there are n desserts in total, there are n! ways to arrange them in the order of preference.

Combinations - order does not matter

Combinations can be used when we need to choose **k** objects from a set of **n** objects without repetition("replacement").

The formula for combinations is: ${}_nC_k = \dfrac{n!}{k!(n-k)!}$

Notice how the formula for combinations is similar to permutations, we additionally divide by k! which is the number of ways to arrange k objects because in combinations order does not matter (all the different arrangements count as 1)

Pascal's Triangle

Today we discussed the following problem:

How many ways are there to go from the bottom left corner of the chessboard to the upper right, moving always only to the right and up?

This leads us to the following table (we only show part of it):

1	6	21	56	126	252
1	5	15	35	70	126
1	4	10	20	35	56
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1

These numbers are called the *binomial coefficients*. They are usually written in a slightly different way:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \dots & & & & \end{array}$$

This triangle is called *Pascal triangle*. Every entry in it is obtained as the sum of two entries above it. The k-th entry in n-th line is denoted by $\binom{n}{k}$, or by ${}_nC_k$. Note that both n and k are counted from 0, not from 1: for example, $\binom{2}{1} = 2$.

Problems

1. Finish the chessboard problem: how many ways are there to go from lower left corner to upper right corner?

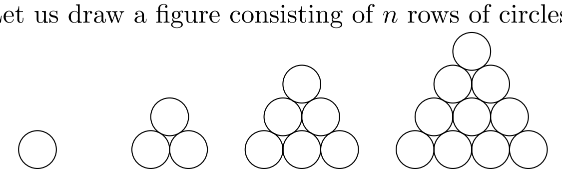
2. Which of the numbers in Pascal triangle are even? Can you guess the pattern, and then carefully explain why it works?

3. What is the sum of all entries in the nth row of Pascal triangle? Try computing first several answers and then guess the general formula.

4. What is the alternating sum of all the numbers in nth row of Pascal triangle, i.e.

$$1 - {}_nC_1 + {}_nC_2 - {}_nC_3 + \dots$$

5. Let us draw a figure consisting of n rows of circles as shown in the figure below (for $n = 1, 2, 3, 4$):



Let T_n be the number of circles in nth figure (for example, $T_1 = 1$, $T_2 = 3$, $T_3 = 6\dots$). These numbers are sometimes called the triangular numbers.

(a) What is the difference $T_{n+1} - T_n$?

(b) Show that the numbers T_n appear in the Pascal triangle as shown below

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & \boxed{1} & & \\ & 1 & & 3 & & \boxed{3} & & 1 & \\ 1 & & 4 & & \boxed{6} & & 4 & & 1 \\ & & & & \dots & & & & \end{array}$$

(that is, $T_n = {}_{n+1}C_2$)

6. A dinner in a restaurant consists of 3 courses: appetizer, main course, and dessert. There are 5 possible appetizers, 6 main courses and 3 desserts. How many possible dinners are there?

7. How many ways are there to seat 5 students in a class that has 5 desks? if there are 10 desks?

8. How many ways are there to select first, second and third prize winner if there are 14 athletes in a competition?

9. How many ways are there to put 8 rooks on a the chessboard so that no one attacks the others?