MATH 10
ASSIGNMENT 14: LIMITS
JANUARY 14, 2024

Limits

Let X be a metric space and let a_n be a sequence of points in X.

We say that a sequnce a_n has limit A if, as n increases, terms of the sequence get closer and closer to A.

This definition is not very precise. For example, the terms of sequence $a_n = 1/n$ get closer and closer to 0, so one expects that the limit is 0. On the other hand, it is also true that they get closer and closer to -1. So the words "closer and closer" is not a good way to express what we mean.

A better way to say this is as follows.

Definition. A set U is called a *trap* for the sequence a_n if, starting with some index N, all terms of the sequence are in this set:

$$\exists N: \quad \forall n \geq N: \ a_n \in U$$

Note that it is not the same as "infinitely many terms of the sequence are in this set".

Now we can give a rigorous definition of a limit.

Definition. A point $A \in X$ is called the *limit* of sequence a_n (notation: $A = \lim a_n$) if for any $\varepsilon > 0$, the neighborhood $B_\varepsilon(A) = \{x \mid d(x, A) < \varepsilon\}$ is a trap for the sequence a_n.

For example, when we say that for a sequence of real numbers a_n we have $\lim a_n = 3$, it means:

there is an index N such that for all $n \geq N$ we will have $a_n \in (2.99, 3.01)$,
there is an index N' (possibly different) such that for all $n \geq N'$ we will have $a_n \in (2.999, 3.001)$
there is an index N'' such that for all $n \geq N''$ we will have $a_n \in (3 - 0.0000001, 3 + 0.0000001)$
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1. Consider the sequence $a_n = 1/n$.
   (a) Fill in the blanks in each of the statements below so that they become true statements:
   – For all $n \geq$ ____, $|a_n| < 0.1$
   – For all $n \geq$ ____, $|a_n| < 0.001$
   – For all $n \geq$ ____, $|a_n| < 0.00017$
   (b) Show that $\lim a_n = 0$.

2. Prove that $\lim \frac{1}{n(n+1)} = 0$ (hint: $\frac{1}{n(n+1)} < \frac{1}{n}$).

3. Find the limits of the following sequences if they exist:
   (a) $a_n = \frac{1}{n^2}$
   (b) $a_n = \frac{1}{2^n}$
   (c) $a_n = n$

4. Explain why the number 1 is NOT a limit of the sequence $(-1)^n$.

5. (a) Show that if a sequence of real numbers has limit $\lim a_n = -1$, then starting with some index, all terms of this sequence are negative.
   (b) Show that it is impossible for this sequence to also have limit $\lim a_n = 1$.

6. (a) Show that if all terms of a sequence of real numbers are non-negative, then its limit (if exists) is also non-negative.
   (b) Is it true that if all terms of a sequence are positive, then its limit (if exists) is also positive?

7. Let $\lim a_n = A$, and let U be an open set containing A. Show that then, starting with some index N, all terms of the sequence a_n are in U.

8. Let $S \subset X$ be a closed set. Let a_n be a sequence such that for all n, $a_n \in S$ and which has a limit. Show that then $\lim a_n \in S$. [Hint: let S' be the complement of S. Then S' is open, so every point of S' is an interior point...]

9. Show that the limit of a sequence, if exists, is unique: it is impossible that $\lim a_n = A$ and also $\lim a_n = A'$, with $A \neq A'$. [Hint: A, A' have non-intersecting neighborhoods.]