MATH 10
ASSIGNMENT 8: CROSS-PRODUCT
NOVEMBER 12, 2023

Signed area: review

Recall that we had defined "wedge product" of two vectors in the plane by

$$\mathbf{v} \wedge \mathbf{w} = x_1 y_2 - y_1 x_2 \in \mathbb{R} \tag{1}$$

One can think of $\mathbf{v} \wedge \mathbf{w}$ as "signed area":

$$\mathbf{v} \wedge \mathbf{w} = \begin{cases} S_{ABCD}, & \text{if rotation from } \mathbf{v} \text{ to } \mathbf{w} \text{ is counterclockwise} \\ -S_{ABCD}, & \text{if rotation from } \mathbf{v} \text{ to } \mathbf{w} \text{ is clockwise} \end{cases}$$

The wedge product (and thus, the signed area) is in many ways easier than the usual area. Namely, we have:

1. It is linear: $(\mathbf{v}_1 + \mathbf{v}_2) \wedge \mathbf{w} = \mathbf{v}_1 \wedge \mathbf{w} + \mathbf{v}_2 \wedge \mathbf{w}$
2. It is anti-symmetric: $\mathbf{v} \wedge \mathbf{w} = -\mathbf{w} \wedge \mathbf{v}$

Cross-product

If $\mathbf{v}, \mathbf{w}$ are two vectors in $\mathbb{R}^3$, then we can define a different kind of product, called the cross-product, which is a **vector** in $\mathbb{R}^3$, defined by

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

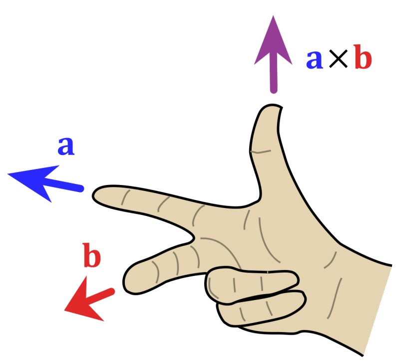
For example, if $\mathbf{v}$, $\mathbf{w}$ are vectors in the xy plane, then $\mathbf{v} \times \mathbf{w}$ is a vector along the direction of the z-axis; moreover, in this case

$$\mathbf{v} \times \mathbf{w} = (\mathbf{v} \wedge \mathbf{w})\mathbf{j}$$

where $\mathbf{j}$ is the unit vector in the positive direction of z-axis.

The cross-product has several important properties. Some of them are proved below, others are left without a proof.

1. It is linear in $\mathbf{v}$, $\mathbf{w}$: $(\mathbf{v}' + \mathbf{v}'') \times \mathbf{w} = \mathbf{v}' \times \mathbf{w} + \mathbf{v}'' \times \mathbf{w}$, and similarly for $\mathbf{w}$
2. It is anti-symmetric: $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$
3. $|\mathbf{v} \times \mathbf{w}|$ = area of the parallelogram with sides $\mathbf{v}$, $\mathbf{w}$
4. $\mathbf{v} \times \mathbf{w}$ is perpendicular to the plane containing $\mathbf{v}$, $\mathbf{w}$
5. The direction of $\mathbf{v} \times \mathbf{w}$ is determined by so-called right hand rule:



Thus, if $\mathbf{v}$ is along positive direction of x axis, and $\mathbf{w}$ is in the positive direction of y-axis, then $\mathbf{v} \times \mathbf{w}$ will be in the positive direction of the z-axis.

1. Check that if $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along positive directions of x, y, z axes respectively, then

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

and similar for the cyclic permutations of these three vectors: $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

2. (a) Use cross-product to construct a vector perpendicular to both of the vectors below:

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) Write the equation of the plane through points $(0, 0, 0)$, $(1, 0, 2)$, $(1, 1, 1)$.

3. Show that if all vertices of a triangle in the plane have integer coordinates, then its area A is a half-integer (i.e., $2A \in \mathbb{Z}$). Is the same true for any polygon?

4. For three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$, define the triple product $T(\mathbf{u}, \mathbf{v}, \mathbf{w})$ by the formula

$$T(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{v} \times \mathbf{w}) \bullet \mathbf{u}$$

(note that it is a number, not a vector). The notation T is not standard.

(a) Write an explicit formula the triple product in terms of x, y, and z components of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

(b) Check that the triple product is linear in each of the three vectors and is anti-symmetric:

$$T(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -T(\mathbf{v}, \mathbf{u}, \mathbf{w})$$

and similarly for any other transposition (interchange of any two of the three vectors).

(c) Deduce from part (b) that $T(\mathbf{u}, \mathbf{v}, \mathbf{w}) = T(\mathbf{w}, \mathbf{u}, \mathbf{v})$. [Hint: one can get triple $\mathbf{w}, \mathbf{u}, \mathbf{v}$ from $\mathbf{u}, \mathbf{v}, \mathbf{w}$ by two transpositions]

(d) Deduce from part (b) that $T(\mathbf{v}, \mathbf{v}, \mathbf{w}) = 0$ and thus $\mathbf{v} \times \mathbf{w}$ is perpendicular to $\mathbf{v}$.

(e) Show that for a parallelepiped P with edges given by vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, its volume is given by

$$V_P = |T(\mathbf{u}, \mathbf{v}, \mathbf{w})|$$

5. What is the volume of a tetrahedron with vertices $A = (0, 0, 0)$, $B = (x_1, y_1, z_1)$, $C = (x_2, y_2, z_2)$, $D = (x_3, y_3, z_3)$?