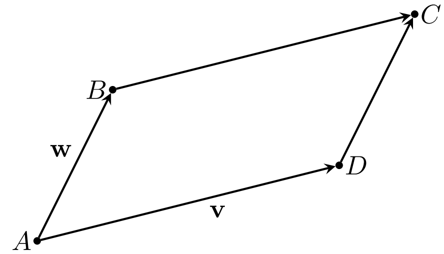
MATH 10
ASSIGNMENT 7: SIGNED AREA
NOV 3, 2023

Signed area

Let $ABCD$ be a parallelogram on the plane, with vertex A at the origin and vertices $D = (x_1, y_1)$, $B = (x_2, y_2)$, so that its sides are vectors

$$\mathbf{v} = \vec{AD} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad \mathbf{w} = \vec{AB} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$



In this case, the area of the parallelogram can be computed as follows:

$$S_{ABCD} = |x_1 y_2 - y_1 x_2| \tag{1}$$

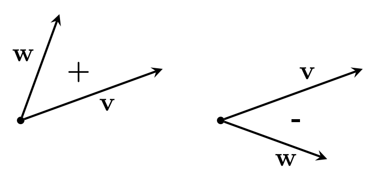
(we will prove this in problem 1 below). We will introduce a new kind of "product" for two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^2$ by

$$\mathbf{v} \wedge \mathbf{w} = x_1 y_2 - y_1 x_2 \in \mathbb{R} \tag{2}$$

if $\mathbf{v}, \mathbf{w}$ are as above (symbol $\wedge$ reads "wedge"). Thus, $S_{ABCD} = |\vec{AD} \wedge \vec{AB}|$. Note that as with the dot product, the wedge product is a number, not a vector.

One can think of $\mathbf{v} \wedge \mathbf{w}$ as "signed area":

$$\mathbf{v} \wedge \mathbf{w} = \begin{cases} S_{ABCD}, & \text{if rotation from } \mathbf{v} \text{ to } \mathbf{w} \text{ is counterclockwise} \\ -S_{ABCD}, & \text{if rotation from } \mathbf{v} \text{ to } \mathbf{w} \text{ is clockwise} \end{cases}$$



The wedge product (and thus, the signed area) is in many ways easier than the usual area. Namely, we have:

1. It is linear: $(\mathbf{v}_1 + \mathbf{v}_2) \wedge \mathbf{w} = \mathbf{v}_1 \wedge \mathbf{w} + \mathbf{v}_2 \wedge \mathbf{w}$
2. It is anti-symmetric: $\mathbf{v} \wedge \mathbf{w} = -\mathbf{w} \wedge \mathbf{v}$

1. The goal of this problem is to give a careful proof of formula (1).

(a) Show that $S_{ABCD} = |\mathbf{v} \bullet R(\mathbf{w})|$, where R is the operation of rotating by 90° clockwise. [Hint: $S = |\mathbf{v}|\,|\mathbf{w}|\,\sin(\varphi)$.]

(b) Show that for a vector $\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix}$, we have $R(\mathbf{w}) = \begin{bmatrix} y \\ -x \end{bmatrix}$.

(c) Deduce from this formula (1).

2. Let $\mathbb{S}_{ABC}$ be the signed area of triangle ABC:

$$\mathbb{S}_{ABC} = \begin{cases} S_{ABC} & \text{if vertices } A,\ B,\ C \text{ go in counterclockwise order} \\ -S_{ABC} & \text{if vertices } A,\ B,\ C \text{ go in clockwise order} \end{cases}$$

Note that $\mathbb{S}_{ABC}$ depends not just on the triangle but also on the order in which we list the vertices. Show that

$$\mathbb{S}_{ABC} = \tfrac{1}{2}\vec{AB} \wedge \vec{AC}.$$

3. Find the area of the triangle with vertices at $(0,0)$, $(5,1)$, $(7,7)$.

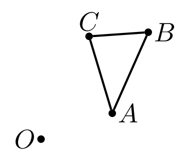
4. If the area of $\triangle ABC$ is 24, what is the area of $\triangle ABM$, where M is the intersection point of the medians?

[This problem can be solved in many ways. One of them: if $\vec{AB} = \mathbf{v}$, $\vec{AC} = \mathbf{w}$, then what is $\vec{AM}$?]

5. Shoelace formula.

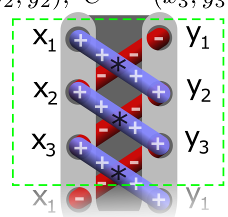
(a) Consider a triangle ABC in the plane; let $\mathbb{S}_{ABC}$ be as in problem 2. Show that then for any point O in the plane, we have

$$\mathbb{S}_{ABC} = \mathbb{S}_{OAB} + \mathbb{S}_{OBC} + \mathbb{S}_{OCA} = \mathbb{S}_{OAB} + \mathbb{S}_{OBC} - \mathbb{S}_{OAC}$$



Note that there are many possible configurations: for example, O could be on the other side of BC, or it could be inside ABC. Do you think the above formula holds in all configurations or only in some?

(b) Consider triangle ABC, where $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$. Show that then,



$$\mathbb{S}_{ABC} = \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1),$$
$$S_{ABC} = \tfrac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1|$$

[Hint: use the previous part with $O = (0,0)$.]

*(c) Can you suggest an analog of this formula for a quadrilateral? for an n-gon?

(d) Find the area of the quadrilateral with vertices at $(1,3)$, $(1,1)$, $(2,1)$, and $(2021, 2022)$.