ADVANCED MATH PROBLEM SOLVING CLUB
EULER'S THEOREM

APRIL 21, 2024

Planar graphs

A *planar graph* is a graph drawn in the plane so that edges do not intersect (except at vertices). Note that the edges need not be straight lines: they can be curved.

Not every graph can be drawn so. For example, it is known that a graph consisting of 6 vertices (3 red, 3 blue) so that each red vertex is connected to each of the blue vertices by one edge, is not a planar graph. [See problem 8 below.]

A planar graph divides the plane into a number of regions (faces).

Euler's formula

For a connected planar graph, the numbers of vertices, edges, and faces are related by Euler's formula:

$$F - E + V = 1$$

(F is the number of finite regions; the region outside the graph is not included in the count.)

In particular, if the graph is a tree (i.e., there are no loops — between any two vertices there is a unique path), then $F = 0$, so that $V - E = 1$.

Problems

1. Show that for any connected graph, it is possible to remove some edges so that the resulting graph is a tree. [It is called a spanning tree of the graph; it is not unique.]

2. Volleyball net is a 1m by 9m rectangle. Each square of the net is 10×10 cm. What is the maximal number of cuts one can make in the net before it falls apart and becomes disconnected?

3. What is the analog of Euler's formula for graphs on the sphere? On the surface of a torus?

4. (a) Is it possible to make a ball by sewing together hexagonal pieces, so that at each corner exactly three pieces meet? The pieces need not be regular hexagons, but each must be a hexagon: six sides, six vertices. [Hint: if it were possible, we would get a graph on the sphere. If it has n faces (hexagons), how many edges and vertices would it have?]

(b) If we allow both hexagons and pentagons, what is the smallest number of pentagons one must use?

After you did this problem, check out this website: http://pub.ist.ac.at/~edels/hexasphere/

5. It is known that there exists a solid body all of whose faces are triangles, and at each vertex, exactly 5 triangles meet. How many faces does it have?

6. Show that in a planar graph without multiple edges (i.e., in which between any two vertices there is at most one edge), we have $2E > 3F$. [Hint: each region has at least 3 edges bounding it...]

7. Show that the graph K_5, consisting of 5 vertices, each connected to each other by exactly one edge, is not planar. [Hint: use Eulers formula.]

8. Show that the graph $K_{3,3}$, consistng of 3 red and 3 blue vertices, each red vertex connected to each blue vertex by one edge, is not planar. [Hint: if it were, each face would be bounded at least 4 edges.]

Can such a graph be drawn on a torus without self-intersections?

9. A graph is called 2-colorable if it is possible to color the vertices using two colors so that endpoints of each edge are of different colors.

(a) Show that every tree is 2-colorable.

(b) Show that for any connected graph, it is possible to remove some of its edges so that the resulting graph is connected and 2-colorable.

*(c) Show that for any connected graph, it is possible to remove at most half of its edges so that the resulting graph is connected and 2-colorable.

10. In a certain country there are 100 cities, some of which are connected by airline routes. It is known that it is possible to fly from any city to any other (possibly with some stops).

Prove that it is possible to visit all cities by making at most 196 flights.

[Hint: see problem 1.]

***11.** For which n and k there exists a planar graph with n vertices, each connected to exactly k other vertices?

There is no simple answer to this problem — it is a mini research problem. Some cases are easty to study, some are hard. Try to see how much you can do. For example, if we take $k = 3$, for which n such a graph exists?