MATH CLUB: RECURRENT SEQUENCES

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In many problems, a sequence is defined using a recurrence relation, i.e. the next term is defined using the previous terms. By far the most famous of these is the Fibonacci sequence:

$$F_0 = 0, F_1 = F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1} \tag{1}$$

The first several terms of this sequence are below:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34, \ldots$$

1. Let a sequence be defined by the following recurrence relation:

$$a_{n+1} = 2a_n - 1, \qquad a_1 = 3$$

Write several terms of this sequence and try to find a pattern. [Hint: look at $a_n - 1$.] Use induction to prove your guess.

2. Daniel is coming up the staircase of 20 steps. He can either go one step at a time, or skip a step, moving two steps at a time.

In how many ways can he come up the stairs?

["Way" refers to (ordered) sequences of his moves, e.g. 1, 2, 1,1, 2, 2,... ; each number represents by how many steps he moved, and the sum must be equal to 20.]

Hint: again, write a recurrence formula!

3. How many ways are there to write a 10-letter "word" consisting of letters A and B if we do not allow letter B to appear two times in a row? What if we allow for B at most two times in a row?

4. How many ways are there to tile a 2×20 strip of paper by 1×2 dominos?

5. A frog sits at vertex A of triangle ABC. Every minute it jumps to one of adjacent vertices.

How many ways there are for the frog to get to vertex A after n minutes? What is the probability that after n minutes, it will be back to A?

[Hint: let a_n be the number of ways of getting back to A after n jumps, and b_n – number of ways of getting to B after n jumps. Try to get recurrence relation for a_n, b_n. If you can't guess the general solution for this recurrence relation, wait until next time.]

6. Consider several first powers of the number $1 + \sqrt{2}$

$$(1+\sqrt{2})^1 = 1 + \sqrt{2} = \sqrt{2} + \sqrt{1}$$
$$(1+\sqrt{2})^2 = 3 + 2\sqrt{2} = \sqrt{9} + \sqrt{8}$$
$$(1+\sqrt{2})^3 = 7 + 5\sqrt{2} = \sqrt{50} + \sqrt{49}$$

Explore these patterns as follows.

Define integers a_n, b_n by $(1+\sqrt{2})^n = a_n + b_n\sqrt{2}$.

(a) Show that then $(1-\sqrt{2})^n = a_n - b_n\sqrt{n}$

(b) Show that $a_n^2 - 2b_n^2 = (-1)^n$. [Hint: $a_n^2 - 2b_n^2 = (a_n - \sqrt{2}b_n)(a_n + \sqrt{2}b_n)$.]

(c) Find recurrent relations for a_n, b_n.

*(d) Try to find the general formula for a_n, b_n.