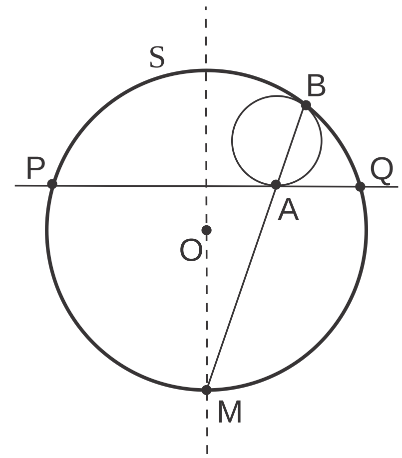
Homework for May 21, 2023.

Geometry.

Review the classwork handout on inversion. Solve the unsolved problems from the previous homework. Solve the exercises and the following problems.

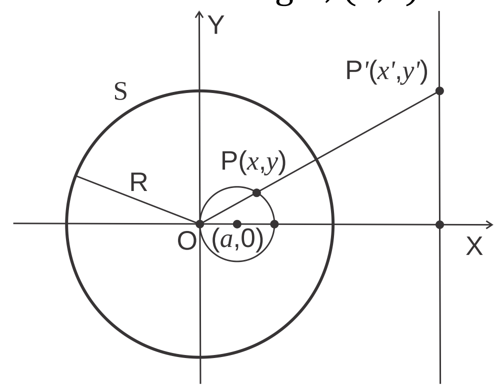
Problems.



1. Consider a circle S with center O and a straight line PQ that cuts from S a circular segment PSQ.
   a. Prove that for any circle inscribed in the segment the line joining the tangency points A and B with the segment and with the circle passes through the midpoint M of the arc PMQ complementary to the segment.
   b. Prove that if two circles inscribed in a circular segment PSQ touch, their common tangent passes through M.
   c. Prove that if two circles inscribed in a circular segment PSQ cross, the line through the two points of intersection passes through M.
   d. A circle overlaps a circular segment so that the four angles it forms with the boundary of the segment are all equal. Let the points of intersection be A_1 and A_2 on the linear segment and B_1 and AB_2 on the arc such that A_1B_2 intersect A_2B_1 inside the segment. Then A_1B_1 and A_2B_2 meet in M.
   e. A circle with center on PQ intersects PQ in A_1 and A_2 and S in B_1 and B_2 (A_1 is inside S, while B_1 is above PQ.) Prove that, if the two cricles meet at 90°, then both A_1B_1 and A_2B_2 pass through M.
2. Steiner's Porism Theorem [Geometry Revisited, p. 124]. Given two circles - one inside the other. Pick up a point in-between and draw a circle tangent to the given two. Then draw a circle tangent to the new circle and the original two. Continue building a chain of circles each touching the two given circles and its predecessor in the chain. It may happen that, for some n, the n-th circle will touch the first circle in the

chain. Prove that if this happens, it will happen regardless of the position of the starting point.

3. Consider inversion with respect to circle S centered at the origin, (0,0). Image of point $P(x, y)$ is point $P'(x', y')$. Prove that the transformation of coordinates is (see figure),



$$x' = x\frac{R^2}{x^2 + y^2}$$
$$y' = y\frac{R^2}{x^2 + y^2}$$

4. What is the image of the line $y = ax + b$?
5. What is the image of a circle $x^2 + y^2 = r^2$?
6. Show that in the case $a \neq r$ there exist x_0, y_0, r_0, such that the image of circle $(x - a)^2 + y^2 = r^2$ is circle $(x' - x_0)^2 + (y' - y_0)^2 = r_0^2$.

Algebra.

Read the classwork handout. Complete the unsolved problems from the previous homework. Solve the following problems. As usual, you do not necessarily have to solve every problem. However, please solve as much as you can in the time you have. Start with those that "catch your eye", in one way or another (e.g. you think are the easiest, or most challenging). Skip the ones you already solved in the past (this is a repeat of the homework on polynomials).

1. Perform long division of the following polynomials.
   a. $(x^5 - 2x^3 + 3x^2 - 4) \div (x^2 - x + 1)$
   b. $(x^4 - x^2 + 1) \div (x + 1)$
   c. $(x^7 + 1) \div (x^3 - x + 1)$
   d. $(6x^6 - 5x^5 + 4x^4 - 3x^3 + 2x - 1) \div (x^2 + 1)$
   e. $(x^5 - 32) \div (x + 2)$
   f. $(x^5 - 32) \div (x - 2)$
   g. $(x^6 + 64) \div (x^2 + 4)$
   h. $(x^6 + 64) \div (x^2 - 4)$
   i. $(x^{100} - 1) \div (x^2 - 1)$
2. Can you find coefficients a, b, such that there is no remainder upon division of a polynomial, $x^4 + ax^3 + bx^2 - 2x - 10$,

a. by $x + 5$
b. by $x^2 + x - 1$

3. Prove that,
   a. for odd n, the polynomial $x^n + 1$ is divisible by $x + 1$
   b. $2^{100} + 1$ is divisible by 17.
   c. $2^n + 1$ can only be prime if n is a power of 2 [Primes of this form are called Fermat primes; there are very few of them. How many can you find?]
   d. for any natural number n, $8^n - 1$ is divisible by 7.
   e. for any natural number n, $15^n + 6$ is divisible by 7
4. Factor (i.e., write as a product of polynomials of smaller degree) the following polynomials.

   a. $1 + a + a^2 + a^3$
   b. $1 - a + a^2 - a^3 + a^4 - a^5$
   c. $a^3 + 3a^2b + 3b^2a + b^3$
   d. $x^4 - 3x^2 + 2$

5. Simplify the following expressions using polynomial factorization.
   e. $\frac{x+y}{x} - \frac{x}{x-y} + \frac{y^2}{x^2-xy}$
   f. $\frac{x^6-1}{x^4+x^2+1}$
   g. $\frac{a^3-2a^2+5a+26}{a^3-5a^2+17a-13}$
6. Solve the following equations
   h. $\frac{x^2+1}{x} + \frac{x}{x^2+1} = 2.9$ (hint: substitution)
   i. $\frac{14}{20-6x-2x^2} + \frac{x^2+4x}{x^2+5x} - \frac{x+3}{x-2} + 3 = 0$ (hint: factorize square polynomials)
7. Write Vieta formulae for the reduced cubic equation, $x^3 + px + q = 0$. Let x_1, x_2 and x_3 be the roots of this equation. Find the following combination in terms of p and q,
   j. $(x_1 + x_2 + x_3)^2$
   k. ${x_1}^2 + {x_2}^2 + {x_3}^2$
   l. $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
   m. $(x_1 + x_2 + x_3)^3$
8. The three real numbers x, y, z, satisfy the equations

$$x + y + z = 7$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{7}$$

Prove that then, at least one of x, y, z is equal to 7. [Hint: Vieta formulas]

9. Find all real roots of the following polynomial and factor it: $x^4 - x^3 + 5x^2 - x - 6$.