April 2, 2023 Math 9

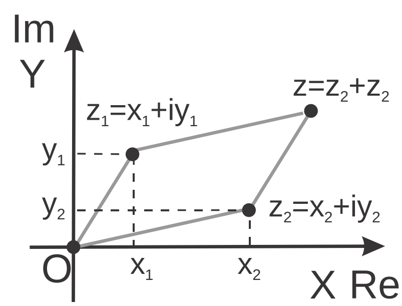
# Complex numbers wrapup. De Moivre formula for negative powers.

$$(\cos\varphi + i\sin\varphi)^{-n} = \cos n\varphi - i\sin n\varphi$$

**Proof.**

$$(\cos\varphi + i\sin\varphi)^{-n} = \frac{1}{(\cos\varphi + i\sin\varphi)^{n}} = \frac{(\cos\varphi - i\sin\varphi)^{n}}{(\cos\varphi + i\sin\varphi)^{n}(\cos\varphi - i\sin\varphi)^{n}}$$
$$= \frac{(\cos(-\varphi) + i\sin(-\varphi))^{n}}{((\cos\varphi + i\sin\varphi)\ (\cos\varphi - i\sin\varphi)\ )^{n}} = \frac{\cos(-n\varphi) + i\sin(-n\varphi)}{(\cos^2\varphi + \sin^2\varphi)^{n}}$$
$$= \cos n\varphi - i\sin n\varphi$$

**Exercise.** Consider the geometrical interpretation of the addition of complex numbers in the figure. Prove that the modulus of the sum of two complex numbers is less or equal the sum of the moduli of these numbers,




$$|z_1 + z_2| \le |z_1| + |z_2|$$

When does the equality hold?

**Exercise.** Without performing calculations, prove that the absolute value of $z = \frac{a+ib}{a-ib}$ is 1.

## Recap: exponential function

The exponential function is usually called a function of the form $f(x) = a^x$ for a special value of the fixed base, $a = e^1 = e \approx 2.71828$, which is a ubiquitous mathematical constant called Euler's number. The exponential function for any other positive real number can be obtained from $e^x$ by using the logarithm. For a real positive $a$, by definition, $a = e^{\log_e a} = e^{\ln a}$, where the logarithm with the base $e$ is called natural logarithm, denoted $f(x) = \log_e x = \ln x$. Hence, an exponential with a base $a$ can be written as, $f(x) = a^x = e^{x\ln a}$. Unless otherwise specified, the term exponential function generally refers to the positive-valued function of a real variable, $x \in \mathbb{R}$, although it can be extended to the complex numbers or generalized to other mathematical objects like matrices or Lie algebras.

The exponential function originated from the notion of exponentiation (repeated multiplication); using the definition of an $n$-th root it is straightforwardly extended to the set of rational numbers. Subsequently, using the monotonic property of an exponential and a construction similar to the Dedekind's section, the definition of an exponential can be rigorously extended to all real values of the argument, $x$.

**Definition.** For an irrational $x \in R$ and $e > 1$, $e^x$ is a number such that that for any rational $q < x$, $e^x > e^q$, while for any rational number $p > x$, $e^x < e^p$,

$$e^x > e^p, \forall p < x, p \in \mathbb{Q}$$
$$e^x < e^p, \forall p > x, p \in \mathbb{Q}$$

Its ubiquitous occurrence in pure and applied mathematics led mathematicians to call the exponential function "the most important function in mathematics". The exponential function can be represented as a power series,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

The exponential function satisfies the following exponential identities,

- $e^x \cdot e^y = e^{x+y}$
- $(e^x)^y = a^{xy}$
- $(e^x)^{\frac{1}{y}} = a^{\frac{x}{y}}$
- $e^0 = 1$

## Exponential of a complex number: Euler's formula

The exponential function of a complex number can be defined such as to satisfy the above exponential identities. Then, for $z = x + iy$,

$$e^z = e^{x+iy} = e^x e^{iy}$$

Since $e^x$ is an exponential of a real number which has already been defined, we only need to define the exponential of a purely imaginary number, $e^{iy}$. Denoting $w = e^{iy}$, we note that, $|w|^2 = w \cdot \overline{w} = e^{iy} \cdot e^{-iy} = e^0 = 1$. Hence,

$$w = e^{iy} = \cos\varphi(y) + i\sin\varphi(y)$$

Here, $\varphi(y)$ is, in general, some function of $y$ whose form should be determined such as to satisfy the exponential identities. Using the exponential identities and the de Moivre formula, we obtain,

$w^n = e^{iny} = (\cos\varphi(y) + i\sin\varphi(y))^n = \cos n\varphi(y) + i\sin n\varphi(y) =$
$\cos\varphi(ny) + i\sin\varphi(ny)$,

wherefrom, $\varphi(ny) = n\varphi(y) + 2\pi k,\ k \in \mathbb{Z}$. If we consider $\varphi(ny)$ to be $0 \le \varphi(ny) < 2\pi$, such that $e^{iny}$ is single-valued, then $\varphi(ny) = n\varphi(y)$. Similarly, for all natural $m$, $\varphi\left(\frac{y}{m}\right) = \frac{1}{m}\varphi(y)$. Finally, from $e^{-iy} = \cos\varphi(y) - i\sin\varphi(y)$ and $e^0 = 1$, it follows that $\varphi(-y) = -\varphi(y)$ and $\varphi(0) = 0$. Hence, we can conclude that for any rational $q$ and any real $y$,

$$\varphi(qy) = q\varphi(y)$$

Applying this equation for the case $y = 1$ we obtain that for any rational value of its argument, $y = q$ the function $\varphi(y)$ is just a linear function,

$$\varphi(q) = q \cdot \varphi(1) = C \cdot q$$

Where $C$ is some constant. If we require that function $\varphi(q)$ is well-behaved, for example is continuous (which means that if the values of its argument are close the values of the function are also close, $\forall\varepsilon > 0, \exists\delta > 0\colon |y_1 - y_2| < \delta \Rightarrow |\varphi(y_1) - \varphi(y_2)| < \varepsilon$), then the above linear property extends to all real values of its argument (this is because the set of rational numbers is dense).

**Exercise.** Prove that $\varphi(y) = C \cdot y, \forall y \in \mathbb{R}$.

In order for $e^{iy} = \cos C \cdot y + i\sin C \cdot y$ to be uniquely defined for $0 \le y < 2\pi$, we need to require $C = 1$. We thus obtain the all-important Euler's formula,

$$e^{iy} = \cos y + i\sin y$$

<u>Homework review.</u>

1. **Problem .** Prove the following equalities:
    a. $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$
    b. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$
    c. $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$
    d. $\sin 4\alpha = 4\sin\alpha\cos^3\alpha - 4\cos\alpha\sin^3\alpha$
    e. $\sin 5\alpha = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha$
    f. $\cos 5\alpha = \cdots$ (find the expression)

**Solution**. Consider the De Moivre formula, for example for

$$(\cos\alpha + i\sin\alpha)^4 = \cos 4\alpha + i\sin 4\alpha$$

Opening the parenthesis on the left we obtain,

$(\cos\alpha + i\sin\alpha)^4 = \cos^4\alpha + 4i\cos^3\alpha\sin\alpha - 6\cos^2\alpha\sin^2\alpha -$
$4i\cos\ \alpha\sin^3\alpha + \sin^4\alpha = \cos^4\alpha - 6\cos^2\alpha\,(1-\cos^2\alpha) + (1-\cos^2\alpha)^2 +$
$4i\cos\ \alpha\sin\ \alpha\,(\cos^2\alpha - \sin^2\alpha) = 1 - 8\cos^2\alpha + 8\cos^4\alpha +$
$4i\cos\ \alpha\sin\ \alpha\,(1 - 2\sin^2\alpha)$,

Wherefrom, equating separately the real and the imaginary parts to the corresponding parts on the right of the De Moivre formula, we obtain,

$$\cos 4\alpha = 1 - 8\cos^2\alpha + 8\cos^4\alpha\,, \qquad \sin\ 4\alpha = 4\cos\ \alpha\sin\ \alpha\,(1 - 2\sin^2\alpha)$$

Similarly, one can obtain the polynomial expressions in $\cos\ \alpha$, $\sin\ \alpha$ for $\cos\ n\alpha$ and $\sin\ n\alpha$.

**2. Trigonometric equations.**
    a. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$

**Solution.** $[\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x]$
$\Leftrightarrow 2\sin 2x\cos x + \sin 2x = 2\cos x\cos 2x + \cos 2x \Leftrightarrow (2\cos x +$
$1)\sin 2x = (2\cos x + 1)\cos 2x \Leftrightarrow \big((\sin 2x = \cos 2x) \vee (2\cos x + 1 =$

$0)\big) \Leftrightarrow \left((\tan 2x = 1) \vee \left(\cos x = -\frac{1}{2}\right)\right) \Leftrightarrow \left(\left(2x = \frac{\pi}{4} + \pi n\right) \vee \left(x = \frac{2\pi}{3} + 2\pi n\right) \vee \left(x = \frac{4\pi}{3} + 2\pi n\right)\right).$

b. $\cos 3x - \sin x = \sqrt{3}(\cos x - \sin 3x)$

**Solution.**

$\left[\cos 3x - \sin x = \sqrt{3}(\cos x - \sin 3x)\right] \Leftrightarrow \left[\frac{1}{2}\cos 3x + \frac{\sqrt{3}}{2}\sin 3x = \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right] \Leftrightarrow \left[\sin\frac{\pi}{6}\cos 3x + \cos\frac{\pi}{6}\sin 3x = \cos\frac{\pi}{3}\sin x + \sin\frac{\pi}{3}\cos x\right]$
$\Leftrightarrow \left[\sin\left(3x + \frac{\pi}{6}\right) = \sin\left(x + \frac{\pi}{3}\right)\right] \Leftrightarrow \left\{\left[3x + \frac{\pi}{6} = x + \frac{\pi}{3} + 2\pi n\right] \vee \left[3x + \frac{\pi}{6} = \pi - \left(x + \frac{\pi}{3}\right) + 2\pi n\right], n \in Z\right\} \Leftrightarrow \left\{\left[2x = \frac{\pi}{6} + 2\pi n\right] \vee \left[4x = \pi - \frac{\pi}{3} - \frac{\pi}{6} + 2\pi n\right], n \in Z\right\} \Leftrightarrow \left\{\left[x = \frac{\pi}{12} + \pi n\right] \vee \left[x = \frac{\pi}{8} + \frac{\pi n}{2}\right], n \in Z\right\}.$

c. $\sin^2 x - 2\sin x\cos x = 3\cos^2 x$

**Solution.**

$[\sin^2 x - 2\sin x\cos x = 3\cos^2 x] \Leftrightarrow \left[1 - 4\cos^2 x = 2\sqrt{1-\cos^2 x}\cos x\right]$
$\Leftarrow [(1 - 4\cos^2 x)^2 = 4(1 - \cos^2 x)\cos^2 x] \Leftrightarrow [20\cos^4 x - 12\cos^2 x + 1 = 0] \Leftrightarrow \left[\cos^2 x = \frac{6 \pm \sqrt{16}}{20}\right] \Leftrightarrow \{[\cos^2 x = 0.5] \vee [\cos^2 x = 0.1]\} \Leftrightarrow \left[\cos x = \pm\frac{\sqrt{2}}{2}\right] \vee \left[\cos x = \pm\frac{\sqrt{10}}{10}\right].$

Out of these 4 solutions, we have to select those which satisfy the original equation, where $1 - 4\cos^2 x$ and $\sin x \cos x$ have the same sign. It is negative for $\cos^2 x = 0.5$ ($\sin x$ and $\cos x$ have different signs) and positive for $\cos^2 x = 0.1$ ($\sin x$ and $\cos x$ have same sign). Therefore, we have two solutions,

$\left\{\left[x = -\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) + 2\pi n\right] \vee \left[x = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) + 2\pi n\right]\right\} \vee \left\{\left[x = \cos^{-1}\left(\frac{\sqrt{10}}{10}\right) + 2\pi n\right] \vee \left[x = \cos^{-1}\left(-\frac{\sqrt{10}}{10}\right) + 2\pi n\right]\right\}$, or,

$\left\{\left[x = -\frac{\pi}{4} + 2\pi n\right] \vee \left[x = \frac{3\pi}{4} + 2\pi n\right]\right\} \vee \left\{\left[x = \cos^{-1}\left(\frac{\sqrt{10}}{10}\right) + 2\pi n\right] \vee \left[x = \pi - \cos^{-1}\left(\frac{\sqrt{10}}{10}\right) + 2\pi n\right]\right\}.$

Finally, this can be recast in the form of an answer,

$\left\{\left[x = -\frac{\pi}{4} + \pi n\right] \vee \left[x = (-1)^n \cos^{-1}\left(\frac{\sqrt{10}}{10}\right) + \pi n\right]\right\}.$