February 25, 2023 Math 9

Geometry. Trigonometry.

Trigonometric formulas and equations.

Using the formulas for the sine and cosine of the sum/difference of two angles, which we have previously derived,

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

It is easy to obtain all other trigonometric formulae.

Exercise. Derive the following expressions for the products of sine and cosine,

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$$

Solution. These expressions are obtained by adding and subtracting the above expressions for $\sin(\alpha \pm \beta)$, $\cos(\alpha \pm \beta)$. For example,

$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$, etc.

Exercise. Derive the following expressions for sums and differences of sine and cosine,

$$\sin\alpha + \sin\beta = 2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right]$$

$$\sin\alpha - \sin\beta = 2\cos\left[\frac{1}{2}(\alpha+\beta)\right]\sin\left[\frac{1}{2}(\alpha-\beta)\right]$$

$$\cos\alpha + \cos\beta = 2\cos\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right]$$

$$\cos\alpha - \cos\beta = -2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\sin\left[\frac{1}{2}(\alpha-\beta)\right]$$

Solution. The above expressions are obtained by representing α and β as, $\alpha = \frac{1}{2}(\alpha+\beta)+\frac{1}{2}(\alpha-\beta)$, $\alpha = \frac{1}{2}(\alpha+\beta)-\frac{1}{2}(\alpha-\beta)$, and using the previously obtained expressions for $\sin(\alpha \pm \beta)$, $\cos(\alpha \pm \beta)$.

Exercise. Derive the following expressions,

$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha)\cos(\beta)} \quad \cot\alpha \pm \cot\beta = \pm\frac{\sin(\alpha \pm \beta)}{\sin(\alpha)\sin(\beta)}$$

$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha) \quad \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \quad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha) \quad \cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1-\cos\alpha)} \quad \cos\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1+\cos\alpha)}$$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$\cot\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{1-\sin\alpha}} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1-\cos\alpha}$$

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$$

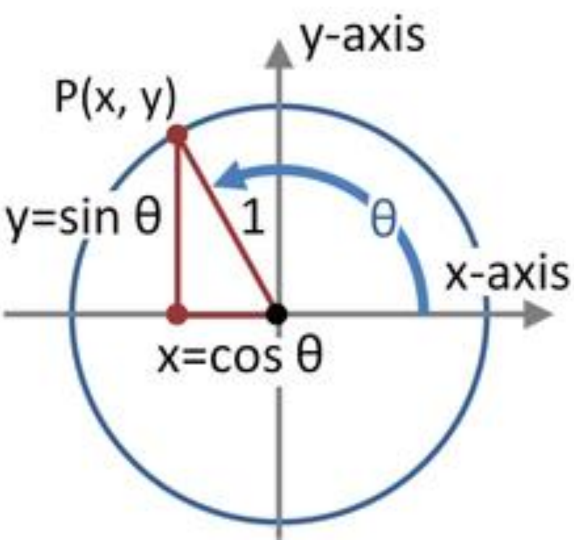
$$\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

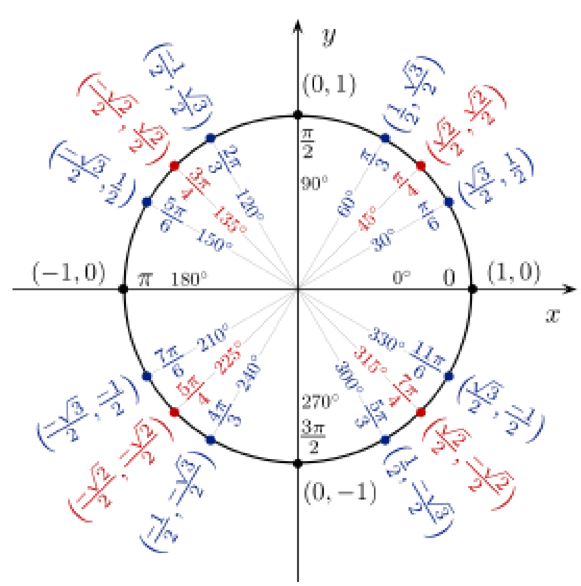
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

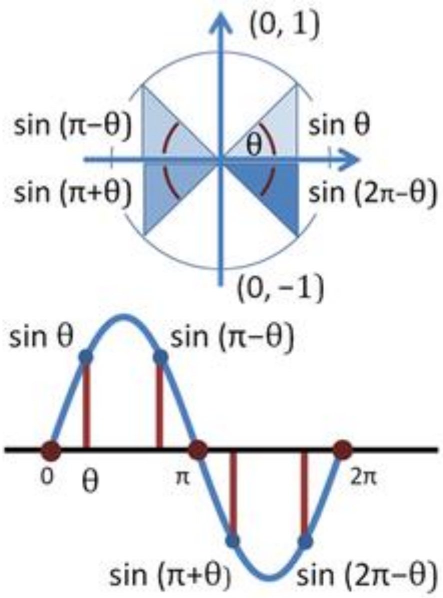
Trigonometric functions and relations.

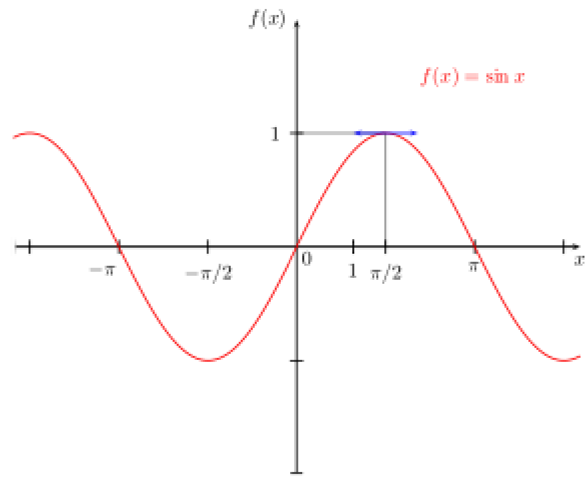
Exercise. Fill in the table of the trigonometric functions of complementary angles below.

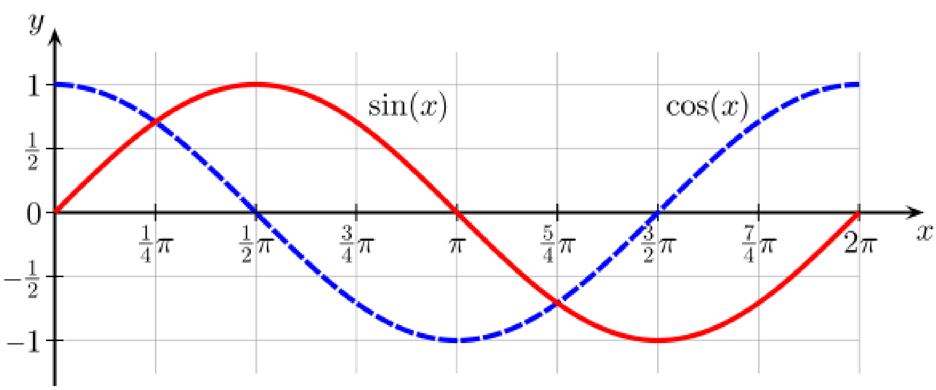
α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$\frac{\pi}{2}-\alpha$	$\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$			
$\frac{\pi}{2}+\alpha$				
$\pi-\alpha$				
$\pi+\alpha$				
$\frac{3}{2}\pi-\alpha$				
$\frac{3}{2}\pi+\alpha$				
$-\alpha$				






(0, 1)
sin (π−θ)
sin θ
θ
sin (π+θ)
sin (2π−θ)
(0, −1)
sin θ
sin (π−θ)
0
θ
π
2π
sin (π+θ)
sin (2π−θ)


f(x)
f(x) = sin x
1
0
−π
−π/2
1
π/2
π
x


y
1
1/2
0
−1/2
−1
sin(x)
cos(x)
1/4 π
1/2 π
3/4 π
π
5/4 π
3/2 π
7/4 π
2π
x