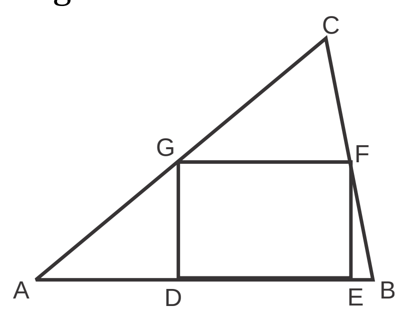
Homework for November 6, 2022.

Geometry.

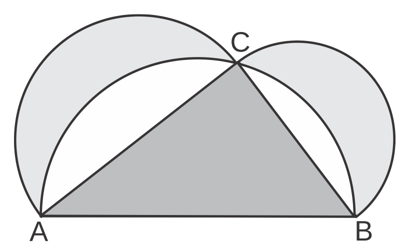
Review the classwork handout. Solve the remaining problems from the previous homework; consider solutions explained in the classwork handout. Try solving the following additional problems. In all the problems, you are only allowed to use theorems we had proven before!

Problems.

1. Find the area and the perimeter of the right triangle such that the lengths of its legs are the roots of the equation, $ax^2 + bx + c = 0$.
2. Rectangle $DEFG$ is inscribed in triangle ABC such that the side DE belongs to the base AB of the triangle, while points F and G belong to sides BC abd CA, respectively. What is the largest area of rectangle $DEFG$?



3. Prove that the area of the two shaded lunes formed by the semi-circles built on the legs and the hypotenuse of the right triangle ABC equals to the area of the triangle ABC (hint: use the generalized Pythagoras theorem).



4. Using a ruler and a compass, construct a circle inscribed in a given angle, AOC (i.e. tangent to both sides, OA and OC of this angle), and a point M inside this angle.
5. Prove that altitudes of any triangle are the bisectors in another triangle, whose vertices are the feet of these altitudes (hint: prove that the line connecting the feet of two altitudes of a triangle cuts off a triangle similar to it).

Algebra.

Review the previous classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems.

1. Using the method of mathematical induction, prove the following equality,

$$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1$$

2. Put the sign $<$, $>$, or $=$, in place of ... below,

$$\frac{n+1}{2} \ldots \sqrt[n]{n!}$$

3. Find the following sum.

$$\left(2+\frac{1}{2}\right)^2 + \left(4+\frac{1}{4}\right)^2 + \cdots + \left(2^n + \frac{1}{2^n}\right)^2$$

4. The lengths of the sides of a triangle are three consecutive terms of the geometric series. Is the common ratio of this series, q, larger or smaller than 2?
5. Solve the following equation,

$\frac{x-1}{x} + \frac{x-2}{x} + \frac{x-3}{x} + \cdots + \frac{1}{x} = 3$, where x is a positive integer.

6. Find the following sum,
   a. $1 + 2 \cdot 3 + 3 \cdot 7 + \cdots + n \cdot (2^n - 1)$
   b. $1 \cdot 3 + 3 \cdot 9 + 5 \cdot 27 + \cdots + (2n-1) \cdot 3^n$
7. Numbers $a_1, a_2, \ldots, a_n$ are the consecutive terms of a geometric progression, and the sum of its first n terms is S_n. Show that,

$$S_n = a_1 a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

8. Prove that three terms shown below are the three terms of the geometric progression, and find the sum of its first n terms, beginning with the first one below,

$$\frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{1}{3-\sqrt{3}}+\frac{1}{6}+\cdots$$

9. What is the maximum value of the expression, $(1+x)^{36}+(1-x)^{36}$ in the interval $|x| \leq 1$?
10. Find the coefficient multiplying x^9 after all parenthesis are expanded in the expression, $(1+x)^9+(1+x)^{10}+\cdots+(1+x)^{19}$.