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Algebra.

Principle of Mathematical Induction.

Let $\{P(n)\} = P(1), P(2), P(3), \dots$ be a sequence of propositions numbered by positive integers, which together constitute a general theorem, P. In particular, $P(n)$ can be some formula, or other property of positive integers. Suppose that by some mathematical argument it can be shown that,

(1) Base Case: $P(1)$ is true, and

(2) Inductive Step: **if** $P(n)$ is true, **then** $P(n+1)$ is true: $P(n) \Rightarrow P(n+1)$.

Then, $P(n)$ is true for all positive integers: $\forall n, P(n)$, so all the propositions of the sequence are true and the theorem P is proved.

The principle of mathematical induction rests on the fact that after any integer, n, there is a next one, $n+1$, and that any integer can be achieved by a finite number of steps incrementing the previous integer by 1, starting from 1.

Although logically obvious, the principle of mathematical induction can be proven as a mathematical theorem using the "principle of smallest integer", which states: **every non-empty set S of positive integers has a smallest number**. Indeed, S must contain at least one integer, say n, and the smallest of integers $1,2,\dots,n$ belonging to S will be the smallest integer in it.

Consider a sequence of statements $\{P(n)\} = \{P(1), P(2), P(3), \dots\}$, such that,

- $P(1)$ is true, and
- For any positive integer if $P(n)$ is true, then $P(n+1)$ is true: $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$.

Let us assume that one of the statements in $P = \{P(n)\}$ is false: $\exists m \in \mathbb{N}, \sim\big(P(m)\big)$ and show that such hypothesis is untenable. Indeed, in such case the set of all positive integers for which $P(n)$ is false is non-empty, and therefore has the smallest number, r. Then, $P(r)$ is false while $P(r-1)$ is true,

$\left(\exists r \in \mathbb{N}, \left(P(r-1) \wedge \sim\left(P(r)\right)\right)\right) \Leftrightarrow \sim(\forall r \in \mathbb{N}, P(r) \Rightarrow P(r+1))$. This contradicts our assumption, which completes the proof. ¤

Let us now recast the above proof using the notations of logical calculus.

<u>**Theorem**</u> (Principle of Mathematical Induction).

$$\left(P(1) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))\right) \Rightarrow (P: \forall n \in \mathbb{N}, P(n)).$$

<u>**Proof**</u>. Assume the opposite. Recalling that, $\sim(Q \Rightarrow P) \Leftrightarrow (Q \wedge \sim P)$, we write, the negation of the above statement as,

$$\left(P(1) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))\right) \wedge \sim(P: \forall n \in \mathbb{N}, P(n)), \text{ or,}$$

$$\left(P(1) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))\right) \wedge (\exists n \in \mathbb{N}, \sim P(n)).$$

Now, using the "principle of smallest integer" we arrive at a contradiction,

$$\left(\exists r \in \mathbb{N}, \left(P(r-1) \wedge \sim\left(P(r)\right)\right)\right) \Leftrightarrow \sim(\forall r \in \mathbb{N}, P(r) \Rightarrow P(r+1)). \text{ ¤}$$

Example 1. Prove that the sum of the n first odd positive integers is n^2,

i.e., $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

Solution. Let $S(n) = 1 + 3 + 5 + \cdots + (2n-1)$.

We want to prove by induction that for every positive integer n, $S(n) = n^2$.

(1) <u>Verify Base Case.</u> For $n = 1$, we have $S(1) = 1 = 1^2$, so the property holds.

(2) <u>Inductive Step.</u> Assume (Induction Hypothesis) that the property is true for a positive integer n, i.e.: $S(n) = n^2$. We must prove that it is also true for $n + 1$, i.e., $S(n+1) = (n+1)^2$, i. e., $\{S(n) = n^2\} \Rightarrow \{S(n+1) = (n+1)^2\}$. In fact, we can verify this explicitly,

$$S(n+1) = 1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = S(n) + (2n+1).$$

But, by induction hypothesis, $S(n) = n^2$. Hence,

$S(n+1) = n^2 + (2n+1) = (n+1)^2$.

This completes the inductive step and shows that the property is true for all positive integers. ¤

Numerical sequences. Progressions.

Numerical sequence is an ordered set of numbers, which are numbered consecutively by positive integers, n, $\{a_1, a_2, a_3, \dots, a_n\}$. The numbers, a_i, are called elements, or terms. The series is the value obtained by adding up all terms in the sequence; this value is called the "sum".

A Series is the sum of the terms of a sequence. Finite sequences and series have both first and last terms defined, whereas infinite sequences and series continue indefinitely.

Arithmetic progression is the following numerical sequence,

$$\{a_1, a_2, a_3, \dots, a_n\} = \{a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \dots a_1 + (n-1)d\}. \qquad (7)$$

The sum of the arithmetic progression is,

$$S_n = \frac{n}{2}(2a_1 + (n-1)d) = \frac{n}{2}(a_1 + a_n). \qquad (8)$$

Exercise. Using mathematical induction, prove that

$$P_n: \sum_{k=1}^{n} k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Solution.

Basis: $P_1: \sum_{k=1}^{1} k = 1 = \frac{1\cdot(1+1)}{2}$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \sum_{k=1}^{n+1} k = 1 + 2 + 3 + \dots + (n+1) = \frac{(n+1)(n+2)}{2}$

Proof: $\sum_{k=1}^{n+1} k = 1 + 2 + 3 + \dots + (n+1) = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$.

__Geometic progression__ is a sequence,

$$\{a_1, a_2, a_3, \dots a_n\} = \{a,\ aq^2, aq^3, \dots, aq^{n-1}\} \tag{9}$$

The sum of the geometric progression is geometric series,

$$G_n = a + aq\ + aq^2 + aq^3 + \cdots +\ aq^{n-1} = a\frac{1-q^n}{1-q}. \tag{10}$$

This can be derived via several methods, including the mathematical induction.

Excercise. Using mathematical induction, prove formula for the sum of geometric series:

$$P_n\colon \sum_{k=0}^{n} q^k\ = 1 + q\ + q^2 + q^3 + \cdots +\ q^n = \frac{1 - q^{n+1}}{1 - q}$$

Solution.

__Basis__: $P_1 : \sum_{k=0}^{1} q^k\ = 1 + q = \frac{1-q^2}{1-q}$

__Induction__: $P_n \Rightarrow P_{n+1}$, where $\sum_{k=0}^{n+1} q^k\ = 1 + q\ + q^2 + q^3 + \cdots +\ q^{n+1} = \frac{1-q^{n+2}}{1-q}$

__Proof__: $\sum_{k=1}^{n+1} q^k\ = 1 + q\ + q^2 + q^3 + \cdots +\ q^n +\ q^{n+1} = \frac{1-q^{n+1}}{1-q} +\ q^{n+1} = \frac{1-q^{n+1}+q^{n+1}-q^{n+2}}{1-q} = \frac{1-q^{n+2}}{1-q}$.

Examples. Using mathematical induction, prove that,

$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3-n}{3}$,

$2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n(2n+1)(n+1)}{3}$.

Formal fallacies (recap). Example of base rate fallacy: Monty Hall Problem.

A formal fallacy is an error in logic that can be seen in the argument's form. All formal fallacies are specific types of *non sequiturs* (does not follow).

- Base rate fallacy – making a probability judgment based on conditional probabilities, without accounting for the effect of prior probabilities.

Example. Consider playing the following game. There is a dollar bill in one of the three boxes. First, you are offered to choose one box. Then, the party you play with randomly chooses to open one of the remaining two boxes and you see that it is empty. You are then offered to swap your box and the remaining un-opened box. Should you switch? In other words, of the two un-opened boxes, the one you have and the unopened box from the other pair, which one has higher probability of containing the dollar?

This is a version of the famous problem in probability theory, also known as Bertrand box paradox, or an extensively discussed Monty Hall Problem, which was popularized by Martin Gardner as the Three Prisoners Problem.

Solution. An incorrect argument, which states that the probabilities of finding the dollar in the two un-opened boxes are equal because they were equal to begin with, before the third box was opened, is an example of base rate fallacy. Discarding the information on the probability of a condition "a box randomly chosen of the remaining two boxes is empty", yields the incorrect conclusion.

One simple solution is to count the possible outcomes: if the dollar is in the box you chose first, then there are two outcomes which satisfy the condition "a box randomly chosen of the remaining two boxes is empty". If, on the other hand, the dollar is in one of those boxes, then there is only one such outcome. Hence, after you know that one of the remaining two boxes is empty, you know that there are twice more chances that the dollar is in the box that remains unopened than in the one you chose first. Alternatively, there is one chance out of three that the dollar is in the box you chose, and two out of three that it is in one of the other two boxes. Once one of those two is opened and found empty, these two out of three chances now are that the dollar is in the unopened box. The solution can be formalized using the Bayes theorem on **conditional probability**.

Bayes' theorem (alternatively Bayes' law or Bayes' rule) describes the probability of an event, based on prior knowledge of conditions that might be related to the event,

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

Here A and B are events whose probabilities without regard to each other are $P(A)$ and $P(B) \neq 0$.

- $P(A|B)$ is a conditional probability, the probability of observing event A given that B is true.
- $P(B|A)$ is the probability of observing event B given that A is true.
- $P(A \wedge B)$ is the probability of observing a joint event of both A and B.

Let A be the event "dollar is in the box A that you picked". $P(A) = \frac{1}{3}$ is the respective unconditional probability. The event B is, "of the two remaining boxes, box B, chosen at random is empty". Note that here we must consider events A and B as independent to obtain their probabilities without regard to each other. The unconditional probability of the event B is, $P(B) = \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{1}{2}$. The conditional probability that an empty box B is opened when both B and C are empty, is $P(B|A) = \frac{1}{2}$, and we obtain,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{1}{3}$$

If C is now the event "dollar is in the remaining box C that has not been opened", then $P(C) = \frac{1}{3}$ and $P(B|C) = 1$, so

$$P(C|B) = \frac{P(B|C)P(C)}{P(B)} = \frac{2}{3}$$

Review of selected homework problems.

Problem 3. Solve the following equations:

a. $\frac{x-a}{x-b}+\frac{x-b}{x-a}=2.5$

Solution.

$\left(\frac{x-a}{x-b}+\frac{x-b}{x-a}=2.5\right) \Leftrightarrow [\left((x-a)^2+(x-b)^2=2.5(x-a)(x-b)\right) \wedge (x\neq a) \wedge$
$(x\neq b)] \Leftrightarrow [\left(2x^2-2(a+b)x+a^2+b^2=2.5(x^2-(a+b)x+ab)\right) \wedge$
$(x\neq a) \wedge (x\neq b)] \Leftrightarrow [(0.5x^2-0.5(a+b)x-a^2+2.5ab-b^2=0) \wedge$
$(x\neq a) \wedge (x\neq b)] \Leftrightarrow [(x^2-(a+b)x-2(a-b)^2+ab=0) \wedge (x\neq a) \wedge$
$(x\neq b)] \Leftrightarrow \left[\left(x=\frac{a+b}{2}\pm\sqrt{\frac{1}{4}(a+b)^2+2(a-b)^2-ab}=0\right) \wedge (x\neq a) \wedge\right.$
$\left.(x\neq b)\right] \Leftrightarrow \left[\left(x=\frac{a+b}{2}\pm\sqrt{\frac{9a^2-18ab+9b^2}{4}}=\frac{a+b}{2}\pm 3\frac{a-b}{2}\right) \wedge (x\neq a) \wedge (x\neq b)\right]$
$\Leftrightarrow [(x=2a-b) \vee (x=2b-a) \wedge (a\neq b)]$

d. $\frac{1}{x^2}+\frac{1}{(x+2)^2}=\frac{10}{9}$

Solution.

$\left[\frac{1}{x^2}+\frac{1}{(x+2)^2}=\frac{10}{9}\right] \Leftrightarrow \left[\left(\frac{1}{(y-1)^2}+\frac{1}{(y+1)^2}=\frac{10}{9}\right) \wedge (y=x+1)\right]$
$\Leftrightarrow \left[\left(\frac{(y-1)^2+(y+1)^2}{(y^2-1)^2}=\frac{2(y^2+1)}{(y^2-1)^2}=\frac{10}{9}\right) \wedge (y=x+1)\right] \Leftrightarrow [(9y^2+9=5y^4-$
$10y^2+5) \wedge (y=x+1) \wedge (y^2\neq 1)] \Leftrightarrow [(5y^4-19y^2-4=0) \wedge (y=x+$
$1) \wedge (y^2\neq 1)] \Leftrightarrow \left[\left(y^2=\frac{19\pm\sqrt{19^2+4\cdot 5\cdot 4}}{2\cdot 5}=\frac{19\pm\sqrt{361+80}}{10}=4\right) \wedge (y=x+1) \wedge\right.$
$\left.(y^2\neq 1)\right] \Leftrightarrow [(y=\pm 2) \wedge (x=y-1) \wedge (y^2\neq 1)] \Leftrightarrow [(x=1) \vee (x=3)]$

f. $1+\sqrt{1+x\sqrt{x^2-24}}=x$

Solution.

$\left[1+\sqrt{1+x\sqrt{x^2-24}}=x\right] \Leftrightarrow [\left(1+x\sqrt{x^2-24}=(x-1)^2\right) \wedge (x-1\geq 0)]$
$\Leftrightarrow [\left(x\sqrt{x^2-24}=x^2-2x\right) \wedge (x\geq 1)] \Leftrightarrow [\left(\sqrt{x^2-24}=x-2\right) \wedge (x\geq 1)]$

$\Leftrightarrow\ [(x^2-24=(x-2)^2=x^2-4x+4)\wedge(x\geq 1)]\Leftrightarrow\ [(4x=28)\wedge(x\geq 1)]$
$\Leftrightarrow\ (x=7)$

Problem 4. Simplify expressions:

a. $\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}$

Solution.

Denote $y=\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}$. Then,

$y^2=x+2\sqrt{x-1}+2\sqrt{\left(x-2\sqrt{x-1}\right)\left(x+2\sqrt{x-1}\right)}+x-2\sqrt{x-1}=$
$\left[\left(2x+2\sqrt{x^2-4(x-1)}\right)\wedge(x\geq 1)\wedge\left(x\geq 2\sqrt{x-1}\right)\right]=\left[\left(2x+\right.\right.$
$\left.\left.2\sqrt{(x-2)^2}\right)\wedge(x\geq 1)\right]=\left[\left(\left(2x-2(x-2)\wedge(x\leq 2)\right)\vee\left(2x+2(x-2)\right)\wedge\right.\right.$
$\left.\left.(x\geq 2)\right)\wedge(x\geq 1)\right]=\left[4\wedge(2\geq x\geq 1)\vee\left((4x-4)\wedge(x\geq 2)\right)\right]$

Therefore, $y=\left[\left(2\wedge(2\geq x\geq 1)\right)\vee\left(2\sqrt{x-1}\wedge(x\geq 2)\right)\right]$.

b. $\dfrac{\sqrt{\sqrt{\frac{x-1}{x+1}}+\sqrt{\frac{x+1}{x-1}}-2}}{\sqrt{(x+1)^3}-\sqrt{(x-1)^3}}\left(2x+\sqrt{x^2-1}\right)$

Solution. $y=\dfrac{\sqrt{\sqrt{\frac{x-1}{x+1}}+\sqrt{\frac{x+1}{x-1}}-2}}{\sqrt{(x+1)^3}-\sqrt{(x-1)^3}}\left(2x+\sqrt{x^2-1}\right)=\dfrac{\sqrt{\frac{x+1-2\sqrt{x-1}\sqrt{x+1}+x-1}{\sqrt{x-1}\sqrt{x+1}}}}{\sqrt{(x+1)^3}-\sqrt{(x-1)^3}}\left(2x+\right.$
$\left.\sqrt{x^2-1}\right)=\dfrac{\sqrt{\left(\sqrt{x+1}-\sqrt{x-1}\right)^2}}{\sqrt{\sqrt{x^2-1}}\left(\sqrt{x+1}-\sqrt{x-1}\right)\left(\left(\sqrt{x-1}\right)^2+\sqrt{x-1}\sqrt{x+1}+\left(\sqrt{x+1}\right)^2\right)}\left(2x+\sqrt{x^2-1}\right)=$
$\dfrac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{\sqrt{x^2-1}}\left(\sqrt{x+1}-\sqrt{x-1}\right)}=\left(\dfrac{1}{\sqrt[4]{x^2-1}}\right)\wedge(x>1)$