MATH 8: HANDOUT 14
EUCLIDEAN GEOMETRY 3: TRIANGLE INEQUALITIES.

8. Isosceles triangles

A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

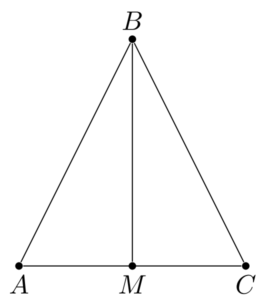
Theorem 11 (Base angles equal). *If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$.*
Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

A proof is given to you as homework.

In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the altitude from A is perpendicular to BC (it exists and is unique by Theorem about the existence of the perpendicular); the median from A bisects BC (that is, it crosses BC at a point D which is the midpoint of BC); and the angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

For general triangle, all three lines are different. However, it turns out that in an isosceles triangle, they coincide.

Theorem 12. *If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.*



Proof. Consider triangles $\triangle ABM$ and $\triangle CBM$. Then $AB = CB$ (by definition of isosceles triangle), $AM = CM$ (by definition of midpoint), and side BM is the same in both triangles. Thus, by SAS axiom, $\triangle ABM \cong \triangle CBM$. Therefore, $m\angle ABM = m\angle CBM$, so BM is the angle bisector.
Also, $m\angle AMB = m\angle CMB$. On the other hand, $m\angle AMB + m\angle CMB = m\angle AMC = 180^\circ$. Thus, $m\angle AMB = m\angle CMB = 180^\circ/2 = 90^\circ$. □

9. Triangle inequalities

In this section, we use previous results about triangles to prove two important inequalities which hold for any triangle.

We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal. The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle.

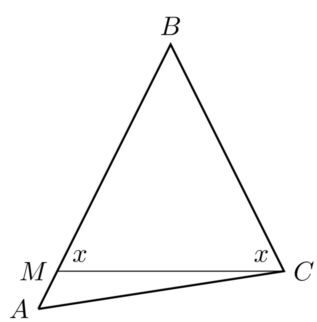
Theorem 13. *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Proof. Assume not. Then either $BC = AB$ or $BC < AB$.

But if $BC = AB$, then $\triangle ABC$ is isosceles, so by Theorem 9, $m\angle A = m\angle C$ as base angles, which gives a contradiction.

Now assume $BC < AB$, find the point M on AB so that $BM = BC$, and draw the line MC. Then $\triangle MBC$ is isosceles, with apex at B. Hence $m\angle BMC = m\angle MCB$ (these two angles are denoted by x in the figure.) On one hand, $m\angle C > x$ (this easily follows from Angle Measurement Axiom). On the other hand, since x is an external angle of $\triangle AMC$, we have $x > m\angle A$ ($x = 180 - \angle CMA = \angle MCA + \angle MAC > \angle MAC$. These two inequalities imply $m\angle C > m\angle A$, which contradicts what we started with.

Thus, assumptions $BC = AB$ or $BC < AB$ both lead to a contradiction.



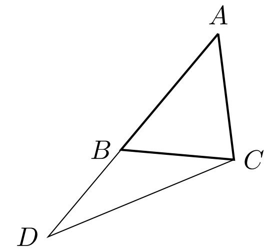
□

The converse of the previous theorem is also true: opposite a longer side, there must be a larger angle.The proof is left as an exercise.

Theorem 14. *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 15 (The triangle inequality)**.** *In $\triangle ABC$, we have $AB + BC > AC$.*



Proof. Extend the line AB past B to the point D so that $BD = BC$, and join the points C and D with a line so as to form the triangle ADC. Observe that $\triangle BCD$ is isosceles, with apex at B; hence $m\angle BDC = m\angle BCD$. It is immediate that $m\angle DCB < m\angle DCA$. Looking at $\triangle ADC$, it follows that $m\angle D < m\angle C$; by Theorem 13, this implies $AD > AC$. Our result now follows from $AD = AB + BD$ (Axiom 2) □

HOMEWORK

Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

1. (Slant lines and perpendiculars) Let P be a point not on line l, and let $Q \in l$ be such that $PQ \perp l$. Prove that then, for any other point R on line l, we have $PR > PQ$, i.e. the perpendicular is the shortest distance from a point to a line.

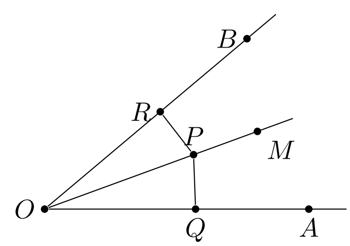
   Note: you can not use the Pythagorean theorem for this, as we haven't yet proved it! Instead, use Theorem 13.

2. (Angle bisector). Define a distance from a point P to line l as the length of the perpendicular from P to l (compare with the previous problem).

   Let $\overrightarrow{OM}$ be the angle bisector of $\angle AOB$, i.e. $\angle AOM \cong \angle MOB$.

   (a) Let P be any point on $\overrightarrow{OM}$, and PQ, PR – perpendiculars from P to sides $\overrightarrow{OA}$, $\overrightarrow{OB}$ respectively. Use ASA axiom to prove that triangles $\triangle OPR$, $\triangle OPQ$ are congruent, and deduce from this that distances from P to $\overrightarrow{OA}$, $\overrightarrow{OB}$ are equal.

   (b) Prove that conversely, if P is a point inside angle $\angle AOB$, and distances from P to the two sides of the angle are equal, then P must lie on the angle bisector $\overrightarrow{OM}$



   These two statements show that the locus of points equidistant from the two sides of an angle is the angle bisector

3. Prove that in any triangle, the three angle bisectors intersect at a single point (compare with the similar fact about perpendicular bisectors)