MATH 8: HANDOUT 10
LOGIC 5: PROOFS CONTINUED

Commonly used laws of logic

- Given $A \implies B$ and A, we can conclude B *(Modus Ponens)*
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! It only means that **if** A is true, then so is C.]
- Given $A \wedge B$, we can conclude A (and we can also conclude B)
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$ *(Modus Tollens)*
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$ *(De Morgan Law)*
- $(A \implies B) \iff ((\neg B) \implies (\neg A))$ *(Law of contrapositive)*

Note: it is important to realize that statements $A \implies B$ and $B \implies A$ are **not** equivalent! (They are called converse of each other).

Common methods of proof

Proof by cases.

Example: Prove that for any integer n, the number $n(n+1)$ is even.

Proof. If n is integer, it is even or odd. If n is even, then $n(n+1)$ is even (a multiple of even is always even). If n is odd, then $n+1$ is even and thus $n(n+1)$ is even by same reasoning.

Thus, in all cases $n(n+1)$ is even. □

<u>*General scheme:*</u>

Given

$$A_1 \vee A_2$$
$$A_1 \implies B$$
$$A_2 \implies B$$

we can conclude that B is true.

You can have more than two cases.

Note: it is important to verify that the cases you consider cover all possibilities (i.e. that at least one of the statements A_1, A_2 is always true).

Conditional proof.

Example: Prove that if n is even, then n^2 is even.

Proof. Assume that n is even. Then $n^2 = n * n$ is also even, since a multiple of even is even. □

<u>*General scheme*</u>

To prove $A \implies B$, we can

- Assume A
- Give a proof of B (in the proof, we can use that A is true).

This proves $A \implies B$ (without any assumptions).

Proof by contradiction.

Example: Prove that if x is a real root of polynomial $p(x) = 10x^3 + 2x + 15$ (i.e. x is a solution of the equation $10x^3 + 2x + 15 = 0$), then x must be negative.

Proof. Assume that x is not negative, i.e. $x \geq 0$. Then $p(x) = 10x^3 + 2x + 15 \geq 15$, which contradicts the fact that x is a root of $p(x)$. Thus, our assumption can not be true, so x must be negative. □

General scheme

To prove that A is true, assume A is false, and derive a contradiction. This proves that A must be true. *Example:* Here is a problem from the previous homework that can illustrate the proof by contradiction. Given that the following are true:

$$A \vee B$$
$$B \implies \neg C$$
$$C \implies ((\neg A) \vee B)$$

prove that C is false (i.e. prove $\neg C$).

Proof. Assume by contradiction that C is true.

1. By *Modus Tollens*, C and $B \implies \neg C$, we get $\neg B$.
2. $A \vee B$ and $\neg B$ implies A.
3. A and $\neg B$ implies $((\neg A) \vee B)$ is false, that is $\neg((\neg A) \vee B)$.
4. Now, combining $\neg((\neg A) \vee B)$ with $C \implies ((\neg A) \vee B)$, by *Modus Tollens*, we get $\neg C$.

Therefore, assuming that C is true, we concluded that $\neg C$ is true, that is, C is false, which is a contradiction. Therefore, C must be false. □

PROBLEMS

1. The following statement is sometimes written on highway trucks:

   If you can't see my windows, I can't see you.

   Can you write an equivalent statement without using word "not" (or its variations such as "can't").

2. Consider the following statement:

   You can't be happy unless you have a clear conscience.

   Can you rewrite it using the usual logic operations such as $\wedge$, $\vee$, $\implies$? Use letter H for "you are happy" and C for "you have a clear conscience".

   Note: proving this statement is not part of the assignment :).

3. Here is another one of Lewis Carroll's puzzles. As before, (a) write the obvious conclusion from given statements; and (b) justify the conclusion, by writing a chain of arguments which leads to it.
   - No one subscribes to the *Times*, unless he is well educated.
   - No hedgehogs can read.
   - Those who cannot read are not well educated.

   It may be helpful to write each of these as a statement about some particular being X, e.g. "If X is a hedgehog, then X can't read."

4. Prove that for any integer number n, the number $n(n+1)(2n+1)$ is divisible by 3. Is it true that such a number must also be divisible by 6?

   You can use without proof the fact that any integer can be written in one of the forms $n = 3k$ or $n = 3k + 1$ or $n = 3k + 2$, for some integer k.

5. You are given the following statements:

   $$A \wedge B \implies C$$
   $$B \vee D$$
   $$C \vee \neg D$$

   Using this, prove $A \implies C$.

6. A function $f(x)$ is called *monotonic* if $(x_1 < x_2) \implies (f(x_1) < f(x_2))$. Prove that a monotonic function can't have more than one root. *[Hint: use assume that it has at least two distinct roots and derive a contradiction.]*

7. Prove by contradiction that there does not exist a smallest positive real number.