MATH 8: HANDOUT 9
LOGIC 4: PROOFS

PROOFS

We are commonly asked to *prove* something: given a series of statements (assumptions), prove another statement (conclusion).

In the simple case where all the statements are just formulas involving some logic variables and basic operations, one way to do it is by writing a truth table: list all possible combinations of values of variables and verify that in each case where all assumptions are true, the conclusion is also true. However, usually that's not how it is done. Instead, we construct a series of intermediate statements, each of which follows from the previous ones and assumptions.

When constructing these intermediate statements, we can again use truth tables or some common laws of logic listed below.

- Given $A \implies B$ and A, we can conclude B *(Modus Ponens)*
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! It only means that **if** A is true, then so is C.]
- Given $A \wedge B$, we can conclude A (and we can also conclude B)
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$ *(Modus Tollens)*
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$ *(De Morgan Law)*
- $(A \implies B) \iff ((\neg B) \implies (\neg A))$ *(Law of contrapositive)*
- [Proof by cases] Given $A \vee B$, $A \implies C$ and $B \implies C$, we can conclude C

Note: it is important to realize that statements $A \implies B$ and $B \implies A$ are **not** equivalent! (They are called converse of each other).

COMMON METHODS OF PROOF

Conditional proof: To prove $A \implies B$, **assume** that A is true; derive B using this assumption.

Proof by contradiction: To prove A, assume that A is **false** and derive a contradiction (i.e., something which is always false – e.g. $B \wedge \neg B$).

Combination of the above: To prove $A \implies B$, assume that A is true and that B is false and then derive a contradiction.

PROBLEMS

In problems 1, 2, you need to (a) write the obvious conclusion from given statements; and (b) justify the conclusion, by writing a chain of arguments which leads to it. It may help to write the given statements and conclusion by logical formulas (denoting the statements which are used by letters $A, B, \dots$ connected by logical operations $\vee, \wedge, \implies, \dots$).

1. If today is Thursday, then Jane's class has library day. If Jane's class has library day, then Jane will bring home new library books. Jane brought no new library books. Therefore,...

2. If Jack comes home late from school, it means he either had a track meet or a theater club. After a track meet, he comes home very tired. Today he came home late but was not tired. Therefore, ...

3. You probably know Lewis Carroll as the author of *Alice in Wonderland* and other books. What you might not know is that he was also a mathematician very much interested in logic, and had invented a number of logic puzzles. Here is one of them:

You are given 3 statements.

(a) All babies are illogical.

(b) Nobody is despised who can manage a crocodile.

(c) Illogical persons are despised.

Can you guess what would be the natural conclusion from these 3 statements? Can you prove it using some laws of logic?

It might help to write each of them as combination of elementary statements about a given person, e.g. B for "this person is a baby", I for "this person is illogical", etc.

4. Here is another of Lewis Carrol's puzzles.

(a) All hummingbirds are richly colored.
(b) No large birds live on honey.
(c) Birds that do not live on honey are dull in color.

Therefore, . . .

(You may assume that "dull in color" is the same as "not richly colored").

Hint: think of all these as statements about some bird X and rewrite in simpler form, using only basic logic operations. E.g., first statement can be rewritten as "If X is a hummingbird, then X is richly colored".

5. Consider the following statements:

(a) In order to visit NY state, you must take a COVID-19 test.
(b) You can only take COVID-19 test if you have high fever or have been in contact with someone who was diagnosed with COVID-19.

Can you draw and prove the conclusion from these statements?

6. Let A, B, C be logical variables. Given that the following are true:

$$A \vee B$$
$$B \implies \neg C$$
$$C \implies ((\neg A) \vee B)$$

prove that C is false.

7. Prove that if a square of an integer number is even, then the number itself is even. [You can use without proof that every integer number is either even (i.e. can be written in the form $n = 2k$) or odd (i.e. can be written in the form $n = 2k + 1$).]