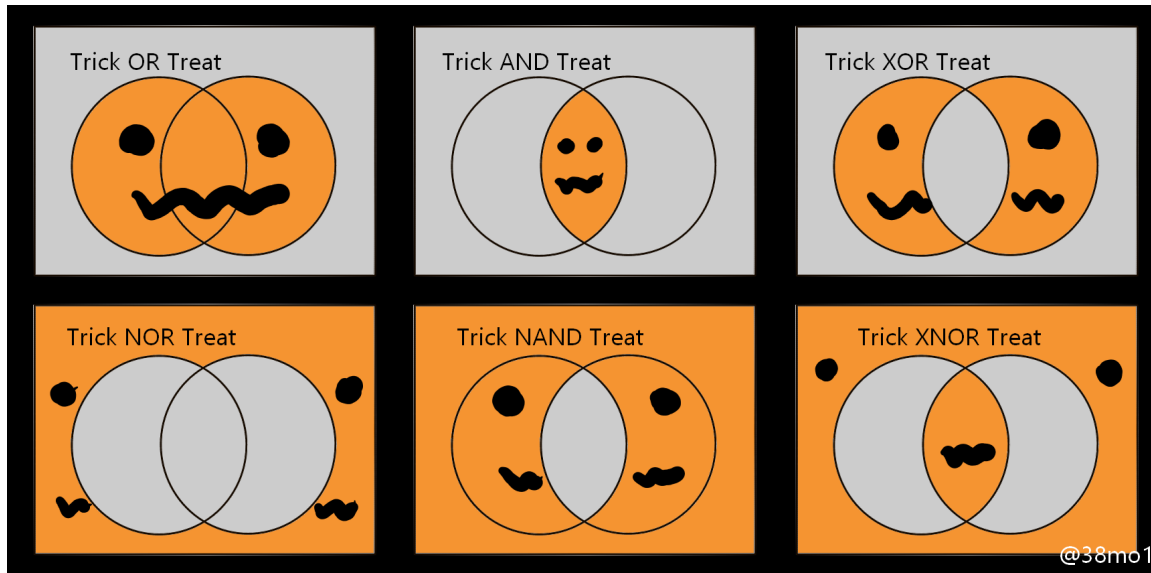


MATH 8: HANDOUT 5
LOGIC 1: INTRODUCTION TO SYMBOLS AND FORMULAS

Today we will start discussing formal rules of logic. In logic, we will be dealing with *boolean* expressions, i.e. expressions which only take two values, TRUE and FALSE. We will commonly use abbreviations T and F for these values.

You can also think of these two values as the two possible digits in binary (base 2) arithmetic: $T = 1$, $F = 0$.

In the usual arithmetic, we have some operations (addition, multiplication, ...) which satisfy certain laws (associativity, distributivity, ...). Similarly, there are logic operations and logic laws.



BASIC LOGIC OPERATIONS

- NOT (for example, NOT A): true if A is false, and false if A is true. Commonly denoted by $\neg A$ or (in computer science) $!A$.
- AND (for example A AND B): true if both A, B are true, and false otherwise (i.e., if at least one of them is false). Commonly denoted by $A \wedge B$
- OR (for example A OR B): true if at least one of A, B is true, and false otherwise. Sometimes also called “inclusive or” to distinguish it from the “exclusive or” described in problem 4 below. Commonly denoted by $A \vee B$.

As in usual algebra, logic operations can be combined, e.g. $(A \vee B) \wedge C$.

TRUTH TABLES

If we have a logical formula involving variables A, B, C, \dots , we can make a table listing, for every possible combination of values of A, B, \dots , the value of our formula. For example, the following is the truth tables for OR and AND:

A	B	$A \text{ OR } B$
T	T	T
T	F	T
F	T	T
F	F	F

A	B	$A \text{ AND } B$
T	T	T
T	F	F
F	T	F
F	F	F

LOGIC LAWS

We can combine logic operations, creating more complicated expressions such as $A \wedge (B \vee C)$. As in arithmetic, these operations satisfy some laws: for example $A \vee B$ is the same as $B \vee A$. Here, “the same” means “for all values of A, B , these two expressions give the same answer”; it is usually denoted by \iff . Here are two other laws:

$$\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$$

$$A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$$

Truth tables provide the most straightforward (but not the shortest) way to prove complicated logical rules: if we want to prove that two formulas are equivalent (i.e., always give the same answer), make a truth table for each of them, and if the tables coincide, they are equivalent.

PROBLEMS

- Write the truth table for each of the following formulas. Are they equivalent (i.e., do they always give the same value)?
 - $(A \vee B) \wedge (A \vee C)$
 - $A \vee (B \wedge C)$.

- Use the truth tables to prove *De Morgan's laws*

$$\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$$

$$\neg(A \vee B) \iff (\neg A) \wedge (\neg B)$$

- Use truth tables to show that \vee is commutative and associative:

$$A \vee B \iff B \vee A$$

$$A \vee (B \vee C) \iff (A \vee B) \vee C$$

Is it true that \wedge is also commutative and associative?

- Another logic operation, called “exclusive or”, or XOR , is defined as follows: $A \text{ XOR } B$ is true if and only if exactly one of A, B is true.
 - Write a truth table for XOR
 - Describe XOR using only basic logic operations AND , OR , NOT , i.e. write a formula using variable A, B and these basic operations which is equivalent to $A \text{ XOR } B$.
- Yet one more logic operation, NAND , is defined by

$$A \text{ NAND } B \iff \text{NOT}(A \text{ AND } B)$$

- Write a truth table for NAND
- What is $A \text{ NAND } A$?

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- Show that you can write $\text{NOT } A$, $A \text{ AND } B$, $A \text{ OR } B$ using only NAND (possibly using each of A, B more than once).

This last part explains why NAND chips are popular in electronics: using them, you can build **any** logical gates.

- A restaurant menu says *The fixed price dinner includes entree, dessert, and soup or salad.*

Can you write it as a logical statement, using the following basic pieces:

E : your dinner includes an entree

D : your dinner includes a dessert

P : your dinner includes a soup

S : your dinner includes a salad

and basic logic operations described above?

- On the island of knights and knaves, there are two kinds of people: Knights, who always tell the truth, and Knaves, who always lie. Unfortunately, there is no easy way of knowing whether a person you meet is a knight or a knave. . .

You meet two people on this island, Bart and Ted. Bart claims, “I and Ted are both knights or both knaves.” Ted tells you, “Bart would tell you that I am a knave.” So who is a knight and who is a knave?