MATH 8: HANDOUT 4
POKER PROBABILITIES. BINOMIAL FORMULA

Poker Probabilities

In the game of poker, a player is dealt five cards from a regular deck with 4 suits ($\spadesuit, \clubsuit, \diamondsuit, \heartsuit$) with card values in the following order: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A. We calculated probabilities of the following combinations:

Royal Flush:: 10, J, Q, K, A of any suit (Example: 10♡, J♡, Q♡, K♡, A♡)
There are only 4 of them.

Straight Flush:: Five cards in a row of the same suit (Example: 6♡, 7♡, 8♡, 9♡, 10♡)
Each of these can start from any card from A to 9, and be in each of the for suits: $9 \times 4 = 36$. Notice that we excluded royal flushes from out computation (if we start with 10, we get a Royal Flush).

Four of a kind:: Four cards of the same value, and one other random card (Example: K♡, K♠, K♢, K♣, 2♣)
Which card $13\times$ Which other value $12\times$ Which other suit $4 = 13 \cdot 12 \cdot 4$.

Full House:: Three cards of the same value, and two cards of the same value (Example: K♡, K♠, K♢, 4♠, 4♣)
Which card for 3 $13\times$ Which three suits $\binom{4}{3}\times$ Which card for a pair $12\times$ Which two suits $\binom{4}{2} = 13\binom{4}{3} \cdot 12\binom{4}{2}$.

Flush:: Five cards of the same suit, not in order (Example: 3♡, 6♡, 8♡, J♡, A♡)
Which suit $4\times$ Which five cards $\binom{13}{5} = 4\binom{13}{5}$. We also need to exclude Royal Flushes and Straight Flushes, so the total is $4\binom{13}{5} - 40$.

Straight:: Five cards in order, possibly of different suits (Example: 5♡, 6♠, 7♢, 8♠, 9♣)
Which card to start from (anything from A to 10) $10\times$ Five suits $4^5 = 10 \cdot 4^5$. From here we also need to exclude Royal Flushes and Straight Flushes, so the final answer if $10 \cdot 4^5 - 40$.

Triple:: Three cards of the same value, and two other random cards (Example: K♡, K♠, K♢, 4♠, 2♣)
Which card $\binom{13}{1}\times$ Which three suits $\binom{4}{3}\times$ Which two other values $\binom{12}{2}\times$ Which two suits for these two random card $4^2 = \binom{13}{1}\binom{4}{3}\binom{12}{2}4^2$.

Two pairs:: Two cards of the same value, two cards of the same value, and a random card (Example: K♡, K♠, 10♢, 10♠, 4♣)
Which two cards $\binom{13}{2}\times$ Two suits for each of pair $\binom{4}{2}^2\times$ Remaining value $11\times$ Remaining suit $4 = \binom{13}{2}\binom{4}{2}^2 \cdot 11 \cdot 4$.

Pair:: Two cards of the same value, and three other random cards (Example: K♡, K♠, Q♢, 4♠, 2♣)
Which card $\binom{13}{1}\times$ Which two suits $\binom{4}{2}\times$ Which three other values $\binom{12}{3}\times$ Which three suits for these three random card $4^3 = \binom{13}{1}\binom{4}{2}\binom{12}{3}4^3$.

To calculate probabilities of each of these combinations, we have to divide the counts above by the total number of poker hands, which is $\binom{52}{5}$. The table below gives the probabilities and odds:

Combination	Count	Probability	Odds
Royal Flush	4	0.000154%	1 : 649,740
Straight Flush	36	0.00139%	1 : 72,192
Four of a Kind	$13 \cdot 12 \cdot 4$	0.024%	1 : 4,165
Full House	$13\binom{4}{3} \cdot 12\binom{4}{2}$	0.1441%	1 : 693
Flush	$4\binom{13}{5} - 40$	0.1965%	1 : 508
Straight	$10 \cdot 4^5 - 40$	0.3925%	1 : 254
Triple	$\binom{13}{1}\binom{4}{3}\binom{12}{2}4^2$	2.1128%	1 : 46.3
Two Pairs	$\binom{13}{2}\binom{4}{2}^2 \cdot 11 \cdot 4$	4.7539%	1 : 20
Pair	$\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3$	42.2569%	1 : 1.37
Nothing		50.1177%	1 : 0.995

MAIN FORMULAS OF COMBINATORICS

We continued studying the numbers $\binom{n}{k}$ from the Pascal triangle – during the last class we figure out that thee numbers answer many various questions:

$$\binom{n}{k} = \text{ The number of paths on the chessboard going } k \text{ steps up and } n-k \text{ to the right}$$
$$= \text{The number of words that can be written using } k \text{ zeros and } n-k \text{ ones}$$
$$= \text{The number of ways to choose } k \text{ items out of } n \text{ if the } \textbf{order does not matter}$$

There exists an explicit formula for them

(1) $$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)!k!}$$

It is possible to understand why this formula works: the numerator $n(n-1)\ldots(n-k+1)$ is a number of permutations of k elements out of n — ${}_nP_k$ (when the order matters). Now, to get the formula for $\binom{n}{k}$, we need to divide the number of permutations by the number of different reorderings of k elements (remember how we divided $n(n-1)$ by 2 when we counted all handshakes in a group of n people?)

BINOMIAL FORMULA

These numbers have one more important application:

(2) $$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n}b^n$$

The general term in this formula looks like $\binom{n}{k} \cdot a^{n-k}b^k$. For example, for $n=3$ we get

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

(compare with the 3rd row of Pascal's triangle)

This formula is called the **binomial formula**; we discussed its proof today.

PROBLEMS

In all the problems, you can write your answer as a combination of factorials, ${}_nC_k$, and other arithmetic – you do not have to do the computations. As usual, please write your reasoning, not just the answers!

1. Use the binomial formula to expand the following expressions:
   (a) $(x-y)^3$
   (b) $(a+3b)^3$
   (c) $(2x+y)^5$
   (d) $(x+2y)^5$

2. Find the coefficient of x^8 in the expansion of $(2x+3)^{14}$

3. Compute $(1+\sqrt{3})^6 + (1-\sqrt{3})^6$

4. Compute $(x+2y)^6 - (x-2y)^6$

5. Show that $(1+\sqrt{3})^{12} + (1-\sqrt{3})^{12}$ is integer.

6. Deduce that Pascal's triangle is symmetric, i.e. $\binom{n}{k} = \binom{n}{n-k}$ in two ways:
   (a) Using the binomial formula for $(x+y)^n$ and $(y+x)^n$.
   (b) Using formula (1).

7. Use the binomial formula to compute
   (a) Sum of all numbers in the n-th row of Pascal's triangle. [Hint: take $a=b=1$ in the binomial formula.]
   (b) Alternating sum of all numbers in the n-th row of Pascal's triangle: $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots$.
   Can you find a way of answerign thsi question without using the binomial formula?

8. Let p be prime.
   (a) Show that each of the binomial coefficients $\binom{p}{k}$, $1 \le k \le p-1$, is divisible by p.
   (b) Show that if a, b are integer, then $(a+b)^p - a^p - b^p$ is divisible by p.

***9.** Long ago, the four nations decided to hold a relay race competition. Forty-eight people signed up, twelve from each of four element-nations: Water, Earth, Fire, Air; however a relay run consists of four people, so only sixteen of those people can compete.

(a) Given that each nation must select four people to form a team, how many ways can this be done?

(b) Now consider they run the competition slightly differently: teams will consist of one person from each nation, and four teams will be chosen. How many ways can this be done?

10. [Some of you may have seen this — but not all of you. You can use a calculator (or Wolfram Alpha) for this problem.]

(a) Given a group of 25 people, we ask each of them to choose a day of the year (non-leap, so there are 365 possible days). How many possible combinations can we get? [Order matters: it is important who has chosen which date]

(b) The same question, but now we additionally require that all chosen dates be different.

(c) In a group of 25 people, what are the chances that no two of them have their birthday on the same day? Conversely, what is the chance that at least two people have the same birthday?