Homework 19: Polynomial inequalities – interval method, quadratic functions - vertex form

HW19 is Due March 8; submit it to Google classroom 15 minutes before the class time.

1. Converting from standard to vertex form

We know how to draw the graph of $y = x^2$. It's a parabola.

- We know that the graph of $y = x^2 + 2$ can be obtained from the graph of $y = x^2$ by shifting up by +2 units (or down, if $y = x^2 - 2$)
- We know that the graph of $y = (x + 5)^2$ can be obtained from the graph of $y = x^2$ by shifting left by 5 units (or right, if $y = (x - 5)^2$).
- Based on the two facts above, we can draw a graph of any function of the type $y = (x - h)^2 + k$. The values (h, k) are the coordinates of the vertex of the parabola.

In general. we can transform any quadratic function $\boldsymbol{y = ax^2 + bx + c}$ to $\boldsymbol{y = a(x - h)^2 + k}$.
This is called transforming from standard into vertex form. The coefficient a has the same value in both forms. Note that, if $b = 0$, then your equation in standard form is already in a vertex form with vertex coordinates $(0, k = c)$.

We will use two ways to convert a quadratic function from standard into vertex form:

- **Method 1: completing the square**. We have learned how to do this using the formulas for fast multiplication. *Example:* $y = 2x^2 + 4x - 2 = 2[\boldsymbol{x^2 + 2x} - 1] = 2\left[x^2 + 2.x.1 + 1^2 - 1^2 - 1\right] = 2[(x + 1)^2 - 1 - 1] = 2[(x + 1)^2 - 2] = 2(x + 1)^2 - 4.$
- **Method 2: find the vertex**. Determine the coefficients a, b, c. Find the vertex *x-coordinate* $\boldsymbol{x_v = h = -\frac{b}{2a}}$.
  Then, substitute $\boldsymbol{x_v}$ in the equation you are converting, and solve for y, $y = a\boldsymbol{x_v}^2 + b\boldsymbol{x_v} + c$.
  The found value is the vertex *y-coordinate,* $\boldsymbol{y_v = k}$.
  Write the equation in a vertex form $y = a(x - h)^2 + k$.

  Example: $y = 2x^2 + 4x - 2, \quad a = 2, \ b = 4, \ c = -2$
  Vertex *x-coordinate:* $x_v = h = -\frac{b}{2a} = -\frac{4}{2.2} = -1$
  Vertex *y-coordinate:* $y_v = 2{x_v}^2 + 4x_v - 2 = 2(-1)^2 + 4(-1) - 2 = 2 - 4 - 2 = -4, \quad k = -4$
  New function: to $y = a(x - \boldsymbol{h})^2 + \boldsymbol{k} = 2(x + 1)^2 - 4$

2. Solving polynomial inequalities using the interval method

The general rules for solving polynomial inequalities (quadratic and higher degree) are as follows:

- Find the roots and factor your polynomial, writing it in the form $p(x) = a(x - x_1)(x - x_2)$ (for a polynomial of degree more than 2, you would have more factors).
- With the roots $x_1; x_2; \dots$: divide the real line into intervals; starting with the first interval, choose a number from that interval to be your x, substitute it in the factored inequality, and determine the sign of each factor in your inequality. Then determine the sign of the product of all factors. Repeat for each interval.
- If the inequality has ≥ or ≤ signs, you should also include the roots themselves in the intervals.
- The intervals whose signs match the sign of the inequality are your solutions.

Example 1. $x^2 + x + 2 > 0$
Solution. We find the roots of the equation $x^2 + x + 2 = 0$ to be $x = -2;\ 1$. The inequality in factored form becomes $(x + 2)(x - 1) > 0$, and the roots $-2, 1$ divide the real line into three intervals $(-\infty\,; -2); (-2; 1); (1; +\infty)$.
It is easy to see that the polynomial $x^2 + x + 2$ is positive on the first and the third intervals and negative on the second one. The solution of the inequality is then all x in intervals one and three ($x < -2$ ***OR*** $x > 1$). We write this also as $x \in (-\infty; -2) \cup (1; +\infty)$. (sign ∪ means "or").
Solving polynomial inequalities of second and higher order makes us realize that if we determine the sign of the first interval, the signs of the following intervals alternate. The graph of the polynomial crosses the $x - axis$ from above ("$+$" interval), goes below ("$-$" interval), … the curve **"snakes"** around the axis when crossing the roots. This is why this method for solving polynomial inequalities is also known as the "snake" method. ***Careful*** – if a factor is raised at even power, the sign will always be positive, and the alternation will not apply.

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer and some justification for why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

1. First, convert the quadratic functions into vertex form using completing the square (Method 1). Then **sketch** each graph's vertex position and its parabola shape next to the function. After that, check if your graph sketch is correct by using a graphing calculator or an online graphing tool and write down if you were correct or wrong.
   a. $y = x^2 + 2x - 3$
   b. $y = x^2 + 2x + 3$
   c. $y = -x^2 + 6x - 9$
   d. $y = 3x^2 + x - 1$

2. Convert the quadratic functions into vertex form using finding the vertex method (Method 2). You must show the steps for finding the vertex x and y coordinates.
   a. $y = x^2 + 2x - 3$
   b. $y = x^2 + 2x + 3$
   c. $y = -x^2 + 6x - 9$
   d. $y = 3x^2 + x - 1$

3. Solve the quadratic equations. Use the roots to convert the inequalities into factored form. Draw the roots on the x-axis and solve the corresponding inequalities using the **interval method**. Write the solutions for the inequalities using intervals.
   Example: Solve $x^2 + 2x - 3 = 0$ find the roots and write the equation as $a(x - x_1)(x - x_2) > 0 \; or < 0$ form. For $x^2 + 2x - 3 > 0$ find the solution in interval form as $x \in$ … . For $x^2 + 2x - 3 < 0$ find the solution in interval form $x \in$ . ..
   Note: these are the same functions as in problems 1 and 2. Sketch again each graph: color with red parts of the graphs where y > 0, and color with blue parts of the graph where y < 0. Please check the class notes about how we did this in class.

   a. $x^2 + 2x - 3 = 0$; $x^2 + 2x - 3 > 0$; $x^2 + 2x - 3 < 0$
   b. $x^2 + 2x + 3 = 0$; $x^2 + 2x + 3 > 0$; $x^2 + 2x + 3 < 0$
   c. $-x^2 + 6x - 9 = 0$; $-x^2 + 6x - 9 > 0$; $-x^2 + 6x - 9 < 0$;
   d. $3x^2 + x - 1 = 0$; $3x^2 + x - 1 > 0$ $3x^2 + x - 1 < 0$

4. **(Optional)** Solve the following inequalities, using the **snake method** (determine the sign of the first interval, then alternate). If the factor $(x - x_n)^n$ is raised <u>at even power</u>, do not switch the sign of the next interval.

   Show the solution on the real line. Write the answer in the interval notation. For each part, graph the polynomial using a graphing tool to see how the graph corresponds to the intervals. You do not have to attach the graph.

   a. $(x-1)(x+2) > 0$
   b. $(x+3)(x-2)^2 \leq 0$
   c. $x(x-1)(x+2) \geq 0$
   d. $x^2(x+1)^5(x+2)^3 > 0$