Math 7: Handout 25
Trigonometry 5: Trigonometric equations [2023/04/23]

We can use our experience solving linear and quadratic equations to solve analogous problems involving trigonometric functions. There are important differences, however. For example, the periodicity of the trigonometric functions usually means that there are many solutions to an equation (an infinite number!). Let's look at a few examples.

SOLVING THE EQUATION $\sin x = \sin c$

Using the trigonometric circle (see figure 1), we see that, for a given c, the solution to the equation $\sin x = \sin c$ is

$$x = c + (2\pi) \times n$$

or

$$x = \pi - c + (2\pi) \times n,$$

where n can be any integer number (that's just because adding full turns doesn't change the sign!), which is why we have an infinite number of solutions. To denote that n can be any (positive or negative) integer number, we write $(\forall n \in \mathbb{Z})$ following the equation. Also, a square bracket ("OR") is a neat way to express that either value satisfies the equation. We can use the same technique to solve equations of the type $\sin x = a$. For example, equation $\sin x = \frac{\sqrt{3}}{2}$ has solutions

$$\left[\begin{array}{l} x = \frac{\pi}{3} + (2\pi) \times n\,, \quad (\forall n \in \mathbb{Z})\,, \\ x = \frac{2\pi}{3} + (2\pi) \times n\,, \quad (\forall n \in \mathbb{Z})\,. \end{array} \right.$$

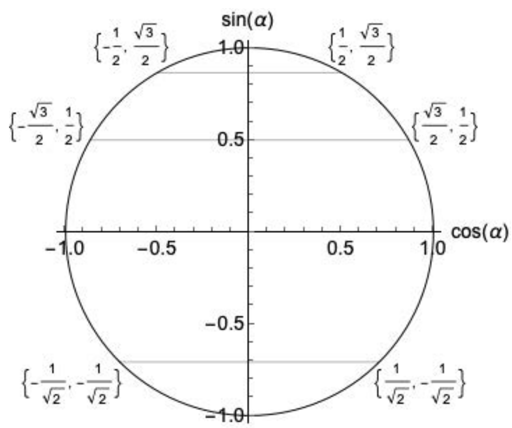

FIGURE 1. A few examples of the relation $\sin x = \sin(\pi - x)$.

SOLVING THE EQUATION $\cos x = \cos c$

By a similar inspection of the trigonometric circle (figure 2), we see that, for a given c, the solution to the equation $\cos x = \cos c$ is

$$\left[\begin{array}{l} x = c + (2\pi) \times n\,, \quad (\forall n \in \mathbb{Z})\,, \\ x = -c + (2\pi) \times n\,, \quad (\forall n \in \mathbb{Z})\,, \end{array} \right.$$

where n can be any integer number. We can use the same technique to solve equations of the type $\cos x = a$. For example, equation $\cos x = \frac{\sqrt{2}}{2}$ has solutions

$$\left[\begin{array}{l} x = \frac{\pi}{4} + (2\pi) \times n\,, \quad (\forall n \in \mathbb{Z})\,, \\ x = -\frac{\pi}{4} + (2\pi) \times n\,, \quad (\forall n \in \mathbb{Z})\,. \end{array} \right.$$

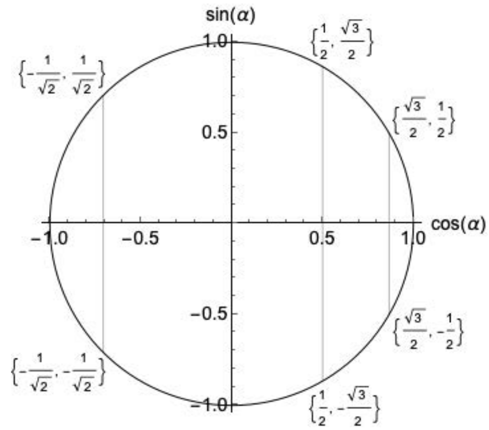

FIGURE 2. A few examples of the relation $\cos x = \cos(-x)$.

GENERAL STRATEGY

We see from these examples that, rather than trying to memorize all different cases, *one should always refer to the trigonometric circle*, from which the answer can be read off immediately. Remember that $\sin$ and $\cos$ are periodic, and there are infinitely many solutions. You can check your answers by comparing the sign and the quadrant,

Quadrant	Range ($\forall n \in \mathbb{Z}$)	$\sin x$	$\cos x$	$\tan x$	$\cot x$
I	$2\pi n < x < 2\pi n + \frac{\pi}{2}$	$+$	$+$	$+$	$+$
II	$2\pi n + \frac{\pi}{2} < x < 2\pi n + \pi$	$+$	$-$	$-$	$-$
III	$2\pi n + \pi < x < 2\pi n + \frac{3\pi}{2}$	$-$	$-$	$+$	$+$
IV	$2\pi n + \frac{3\pi}{2} < x < 2\pi(n+1)$	$-$	$+$	$-$	$-$

HOMEWORK

1. Solve the following equations
- (a) $\sin x = \sin \frac{\pi}{5}$
- (b) $\sin x = 0$
- (c) $\cos x = \frac{\sqrt{2}}{2}$
- (d) $\cos x = \frac{1}{2}$
- (e) $\cos(x + \frac{\pi}{6}) = 0$

2. Solve the following equations
- (a) $(\sin x)^2 - \sin x = 0$ [Hint: start by defining $y = \sin x$.]
- (b) $2(\sin x)^2 - 3\sin x + 1 = 0$
- (c) $4\cos x + \frac{3}{\cos x} = 8$