Math 7: Handout 17 [2023/02/11] :
Absolute values: equations meet inequalities

Equations with absolute values

As you already know, absolute value is a simple function. If $\mathbf{y} = |\mathbf{x}|$, then

$$y = |x| \quad \Leftrightarrow \quad \begin{cases} y = +x & \text{if } x \geq 0\,, \\ y = -x & \text{if } x < 0\,. \end{cases}$$

Things are more interesting if there is more than one absolute value involved. How about function $\mathbf{y} = |\mathbf{x} + \mathbf{2}| + |\mathbf{x} - \mathbf{1}|$? Now there are three regions of x that have to be treated differently:

(a) $x < -2$: both $(x+2)$ and $(x-1)$ are negative, so $y = -(x+2) - (x-1) = -2x - 1$.
(b) $-2 \leq x < 1$: now $(x+2) \geq 0$ and $(x-1) < 0$, so $y = (x+2) - (x-1) = 3$ and independent of x!
(c) Finally, if $x \geq 1$, then both $(x+2)$ and $(x-1)$ are positive, so $y = (x+2) - (x-1) = 2x + 1$.

Why is it important? Let's see how we can solve equation $\mathbf{y} = |\mathbf{x} + \mathbf{2}| + |\mathbf{x} - \mathbf{1}| = \mathbf{7}$.

(a) $x < -2$: then $-2x - 1 = 7 \;\Leftrightarrow\; x = -4$. Since $x = (-4) < (-2)$, this is one of the solutions.
(b) $-2 \leq x < 1$: then $y = 3$ cannot be equal to 7, so there are no solutions in this interval.
(c) $x \geq 1$: then $y = 2x + 1 = 7 \;\Leftrightarrow\; x = 3 > 1$, which is another solution!

The equation has thus two solutions, $\{-4;\, 3\}$, even though it is not a quadratic equation.

The proper way to keep track of all the cases is to use conjunctions (*AND*) and disjunctions (*OR*). Let's solve equation $|\mathbf{2x} - \mathbf{3}| - |\mathbf{x} - \mathbf{1}| = \mathbf{2}$

$$|2x-3| - |x+1| = 2 \;\Leftrightarrow\; \left[\begin{array}{l} \begin{cases} x < -1\,, \\ -(2x-3) + (x+1) = 2\,, \end{cases} \\ \begin{cases} -1 \leq x < \frac{3}{2}\,, \\ -(2x-3) - (x+1) = 2\,, \end{cases} \\ \begin{cases} x \geq \frac{3}{2}\,, \\ (2x-3) - (x+1) = 2\,. \end{cases} \end{array}\right. \;\Leftrightarrow\; \left[\begin{array}{l} \begin{cases} x < -1\,, \\ x = 2\,, \end{cases} \\ \begin{cases} -1 \leq x < \frac{3}{2}\,, \\ x = 0\,, \end{cases} \\ \begin{cases} x \geq \frac{3}{2}\,, \\ x = 6\,. \end{cases} \end{array}\right. \;\Leftrightarrow\; \left[\begin{array}{l} \emptyset\,, \\ x = 0\,, \\ x = 6\,. \end{array}\right.$$

so the equation has two solutions. Note that the first case did not yield a solution because of the constraints we imposed when removed the absolute value sign.

Plotting functions with absolute values

How does one make a graph of function with multiple absolute values, like $\mathbf{y} = |\mathbf{2x} - \mathbf{3}| - |\mathbf{x} + \mathbf{1}|$ above? For each region, $x < -1$, $-1 \leq x < \frac{3}{2}$, and $x \geq \frac{3}{2}$, we had a simple linear formula for the function, so the graph consists of straight-line segments. This is called a *piecewise-linear* function graph. All we have to do is find y values at the points where the graph changes slope. This happens when any of the absolute values in the sum becomes zero:

1. $x = -1$: $y = 5$: to the left of point $(-1, 5)$, $y = -x + 4$, which is a straight line with slope (-1)
2. $x = \frac{3}{2}$: $y = -2\frac{1}{2}$: between points $(-1, 5)$ and $(\frac{3}{2}, -\frac{5}{2})$, $y = -3x + 2$, which is a straight line segment.
3. to the right of point $(\frac{3}{2}, -\frac{5}{2})$, $y = x - 4$, which is a straight line with slope $(+1)$.

This method will work in general. Let's apply it to $\mathbf{y = |x| + |x-1| + |x-3|}$.

1. Find "bend" points where any of the absolute values become zeroes: $x = 0$, $x = 1$, $x = 3|$.
2. Find also the y values at these points: $(x, y) = (0, 4)$, $(1, 3)$, and $(3, 5)$
3. Connect the pairs of points that are next to each other in *horizontal direction*.
4. To the left of the leftmost and to the right of the rightmost point, the graph will be rays extending to infinity. To figure out the slope, you can either use the rules for absolute values *OR* find a point further to the left and to the right, respectively, and draw straight lines. For example, for $x = -1$, $y = 7$: draw a ray starting at $(0, 4)$ and going to the left through point $(-1, 7)$; for $x = 4$, $y = 8$: draw a ray starting at $(3, 5)$ and going to the right through $(4, 8)$.

HOMEWORK

1. Solve these equations

   (a) $|x| + |x+3| = 5$;

   (b) $|x-1| + |3x+5| = 8$.

2. Plot graphs of these piecewise-linear functions

   (a) $y = |x| + |x+3|$;

   (b) $y = |x-1| + |3x+5|$;

   (c) $y = |x-4| + |x-2| + |x+1|$.

   (d) $y = |2x+4| - |x-2| - |x+1|$.

3. You can draw with functions! Write a formula for some function that has the graph shaped like letter "W", and plot it.
   (Use a sum of absolute values as in class).
4. Solve these equations

   (a) $|x^2 - 4x| = |x-2| + 2$;

   (b) $|x^2 + 4x - 5| = |x+2| + 3$.

5. Show that $\sqrt{10 + \sqrt{24} + \sqrt{40} + \sqrt{60}} = \sqrt{2} + \sqrt{3} + \sqrt{5}$.
   Hint: use formula $(a+b+c)^2 = a^2 + (b+c)^2 + 2a(b+c) = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$