Math 7: Handout 13 [2023/01/15] : Vieta Formulas

Vieta formulas

Sometimes we have a problem of finding some expression involving roots of the quadratic equation. Of course, we can always solve the equation, and then do the operations with its roots. However, sometimes it is unnecessary. The formulas below are called **Vieta Formulas** and allow us to find the sum and the product of roots of a quadratic equation without explicitly calculating them.

$$\begin{aligned} x_1 + x_2 &= -\frac{b}{a} \\ x_1 x_2 &= \frac{c}{a} \end{aligned}$$

In a special case if $a = 1$, we have:

$$\begin{aligned} x_1 + x_2 &= -b \\ x_1 x_2 &= c \end{aligned}$$

Why is it so? If an equation $p(x) = 0$ has root x_1 (i.e., if $p(x_1) = 0$), then $p(x)$ is divisible by $(x - x_1)$, i.e. $p(x) = (x - x_1)q(x)$ for some polynomial $q(x)$. Further, if we know both roots – x_1, x_2 — of a quadratic equation $ax^2 + bx + c = 0$, then

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

Example: Quadratic equation $2x^2 - 6x + 2.5 = 0$ has roots $x_1 = 0.5$ and $x_2 = 2.5$ (you can check it using quadratic formula). Then:

$$2x^2 - 6x + 2.5 = 0 = 2(x - 0.5)(x - 2.5).$$

□

Now notice that if $a = 1$, we have the following:

$$x^2 + bx + c = (x - x_1)(x - x_2)$$

Multiplying the expressions in the right hand side, we get:

$$\begin{aligned} x^2 + bx + c &= (x - x_1)(x - x_2) \\ &= x^2 - x_1 \cdot x - x_2 \cdot x + x_1 x_2 \\ &= x^2 - (x_1 + x_2)x + x_1 x_2, \end{aligned}$$

From where we can get the following:

$$\begin{aligned} x_1 + x_2 &= -b \\ x_1 x_2 &= c \end{aligned}$$

Classwork

1. Without solving the quadratic equations, try to guess the roots of equations

(a) $x^2 - 5x + 6 = 0$
(b) $x^2 - x - 2 = 0$
(c) $x^2 - 7x + 12 = 0$
(d) $x^2 - 5x - 6 = 0$

2. Factorize these quadratic expressions

(a) $x^2 + 6x - 7$
(b) $x^2 - 6x + 5$
(c) $x^2 + 3x - 10$
(d) $x^2 - 8x + 12$
(e) $x^4 - 7x - 18$
(f) $x^4 - 17x + 16$

3. (a) Prove that for any $a > 0$, one has $a + \frac{1}{a} \geq 2$, with equality only when $a = 1$.

(b) Show that for any $a, b \geq 0$, one has $\frac{a+b}{2} \geq \sqrt{ab}$.
(The left hand side is usually called the *arithmetic mean* of a, b; the right hand side is called the *geometric mean* of a, b.)

HOMEWORK

1. Let x and y be some numbers. Use the formulas discussed in previous classes to express the following expressions using only $(x + y) = a$ and $xy = b$.
Example: Let us express $x^2 + y^2$ using only $x + y$ and xy. We know that $(x + y)^2 = x^2 + 2xy + y^2$. From here, we get:

$$x^2 + y^2 = (x + y)^2 - 2 \times xy = a^2 - 2 \times b$$

(a) $(x - y)^2$
(b) $\frac{1}{x} + \frac{1}{y}$
(c) $\frac{1}{x-1} + \frac{1}{y-1}$
(d) $x - y$
(e) $x^2 - y^2$
(f) $x^3 + y^3$ (Hint: try finding $(x + y)(x^2 + y^2)$)

2. Let x_1, x_2 be roots of the equation $x^2 + 5x - 7 = 0$. Find

(a) $x_1^2 + x_2^2$
(b) $(x_1 - x_2)^2$
(c) $\frac{1}{x_1} + \frac{1}{x_2}$
(d) $x_1^3 + x_2^3$

3. Solve the following equations:

(a) $x^2 - 9x + 14 = 0$
(b) $x^2 = 1 + x$
(c) $\sqrt{2x + 1} = x$
(d) $x + \frac{1}{x} = 3$

4. Solve the equation $x^4 - 3x^2 + 2 = 0$